

BRST ANALYSIS OF GENERAL MECHANICAL SYSTEMS

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ABSTRACT. We study the groups of local BRST cohomology associated to the general systems of ordinary differential equations, not necessarily Lagrangian or Hamiltonian. Starting with the involutive normal form of the equations, we explicitly compute certain cohomology groups having clear physical meaning. These include the groups of global symmetries, conservation laws and Lagrange structures. It is shown that the space of integrable Lagrange structures is naturally isomorphic to the space of weak Poisson brackets. The last fact allows one to establish a direct link between the path-integral quantization of general not necessarily variational dynamics by means of Lagrange structures and the deformation quantization of weak Poisson brackets.

1. INTRODUCTION

The BRST methods initially appeared as a uniform tool for quantizing either Lagrangian gauge theories or Hamiltonian constrained dynamics (for review see [1]). Correspondingly, the two frameworks have been worked out. The first one, most frequently referred to as a BV or field-anti-field BRST formalism was originally aimed at the problem of covariant path-integral quantization of Lagrangian theories. The second one, commonly called either the BFV formalism or Hamiltonian BRST formalism is most suitable for operator or deformation quantization of the Hamiltonian dynamics. Later on, the BRST formalisms have begun gaining applications in various problems well beyond the original issue of quantization, e.g. in topological field theory [2]. Though the BV and BFV methods share basic principles, they use different prerequisites for constructing the BRST complex and technically are quite different. The relationship between these approaches was established in several ways (see e.g. [3], [4], [5]).

In the recent years, the BRST methods have been extended beyond the scope of Lagrangian or Hamiltonian dynamics [6], [7]. In particular, it was shown that the classical BRST complex can be systematically constructed for a general dynamical system, not necessarily Lagrangian or Hamiltonian. If the dynamical system admits an extra structure, called the *weak Poisson bracket*, then a consistent deformation quantization can be performed in the absence of gauge anomalies [6], [8]. This method can be viewed as a far-reaching extension of the BFV formalism to not necessarily Hamiltonian dynamics. As a prerequisite for the deformation quantisation [6], [8], the

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dynamics should be brought to the *involutive normal form*. This does not restrict the generality, as any regular gauge dynamics can be equivalently formulated in this way [9]. For variational dynamics, the involutive normal form reduces to Dirac's constrained Hamiltonian system, and the corresponding BRST complex [6] reduces to the BFV one.

The corresponding extension of the BV formalism is relied on the new concept of a *Lagrange structure*. Existence of the Lagrange structure is less restrictive for the dynamics than the requirement for the equations to follow from the variational principle. Given a Lagrange structure compatible with equations of motion, the classical theory can be path-integral quantized in several ways [7], [10], [11].

Within the BRST approach, the most of the information about the structure of gauge dynamics is encoded in the groups of local BRST cohomology. In particular, the physical observables, global symmetries, conservation laws, Lagrange structures, quantum anomalies, consistent interactions and counterterms are all the elements of the corresponding cohomology groups. This explains the paramount role that the concept of local BRST cohomology plays in the modern quantum field theory. For Lagrangian theories, several important general theorems on the structure of local BRST cohomology groups were obtained in the last decades of XX century. A comprehensible review of these results can be found in [12], [13] [14]. Recently [15], some of these general theorems were systematically extended beyond the class of Lagrangian dynamics, including the cohomological formulation of Noether's first theorem [16], [17]. It is not surprising that the local BRST cohomology groups, being so informative, are not easy to compute for any nontrivial model, even in Lagrangian setting. More or less complete description of the groups was obtained only for the theories of Yang-Mills type [13], [14]. Some groups have been recently described for the AKSZ-type models in [18].

The present paper is devoted to the study of the groups of local BRST cohomology for mechanical systems whose dynamics are governed by ordinary differential equations (ODEs) of general form. In particular, we do not assume the equations of motion to come from the least action principle that would impose a strong restriction on the structure of dynamics. Since the general theory of ODEs is much more elaborated nowadays than that of PDEs it is reasonable to expect that the corresponding groups of local BRST cohomology are more traceable from the standpoint of computability and physical interpretation. This expectation is generally confirmed by our results below. The main advantage of working with ODEs is the existence of an involutive normal form to which any equations can be locally brought to by introducing auxiliary variables [9]. The procedure of bringing the general dynamics to the involutive form does not impose any restrictions

on dynamics, besides some regularity conditions. For variational systems, the procedure of passing to the involutive normal form reduces to the Dirac-Bergmann algorithm of bringing the general Lagrangian dynamics to the Hamiltonian system with first and second class constraints.

An important advantage of utilizing the involutive normal form is that the BRST charge turns out a local functional on the extended space of trajectories with the integrand involving no more than the first derivatives of dependent variables. It is the absence of higher derivatives which gives an efficient control over the structure of local BRST cohomology and which allows one to bring the calculations of the most interesting groups to the very end. In particular, we give a detailed description for the space of Lagrange structures, which appears to be naturally isomorphic to the space of weak Poisson structures associated with an involutive system of ODEs. In the other words, each Lagrange structure defines a weak Poisson bracket and vice versa. This new fact is important for linking two different branches of the BRST formalism, the BV and BFV ones, in the more general class of dynamics than variational. In particular, it allows one to bridge two different approaches to the quantization of (non-)Lagrangian gauge systems: the path-integral quantization by means of Lagrange structure and the deformation quantization of weak Poisson structure. Although our consideration is restricted to the systems of ODEs, the most of results and computational technique can hopefully be transferred to the field-theoretical models governed by PDEs of evolutionary type. This can require, in principle, a due account of space locality that we do not address in this work. In covariant field theories, however, the space locality is usually related to the locality in time. That is why we can hope that our results on the local BRST cohomology groups, being derived for the systems local in time, will avoid obstructions related to the pure spacial non-locality, at least in the covariant field theories.

Let us also mention some of possible applications of the BRST analysis in the optimal control theory, where the gauge freedom is reinterpreted as the degree of controllability (for an extended discussion see [9], [20]). Among various issues considered in the optimal control there are those concerning isomorphisms of controllable systems and normal forms to which a given controllable system can be brought to by a suitable transformation (static or dynamical feedback equivalence, Lie-Bäcklund isomorphisms, flatness, etc.). The groups of local BRST cohomology, being invariants of all such transformations, provide an efficient tools for attacking these problems. Specifically, one can hope to use them as the spaces of obstructions to global equivalence between two controllable systems and/or as the invariant characterization of normal forms.

The paper is organized as follows. In the next section, we recall the definition of the involutive normal form for a general system of ODEs. For this normal form, we explicitly identify the generators of gauge symmetries and the Noether identities, which are necessary inputs for constructing the classical BRST charge. The classical BRST complex is discussed in Sec. 3. Here we first define the extended symplectic space of trajectories endowed with the Hamiltonian action of the classical BRST charge. We briefly comment on the structure of the classical BRST differential and explain the physical meaning of simplest BRST cohomology groups. Sec. 4 is devoted to computation of the local BRST cohomology groups both in the spaces of local functions and functionals. The computation follows certain systematic procedure. It utilizes a special filtration in the infinite jet spaces that are respected by the Koszul-Tate and longitudinal differentials. This makes possible to work exclusively with functions on finite dimensional spaces and define the corresponding cohomology groups as direct limits. Besides, we intensively exploit the long exact sequences in cohomology (which in our exposition look like exact triangles) and the mapping cone construction. In Sec. 5, we review the construction of the total BRST charge, which is a basic ingredient of the path-integral quantization of (non-)Lagrangian gauge systems. The total BRST charge is defined as a deformation of the classical BRST charge and then reinterpreted as a L_∞ -algebra on a certain space of functionals with the first structure map given by the classical BRST differential. We show that the total BRST charge of an involutive mechanical systems is completely specified by the first and second structure maps. It is the second structure map (weak anti-bracket) which is identified with an integrable Lagrange structure. Sec. 6 establishes a one-to-one correspondence between the spaces of integrable Lagrange structures and weak Hamiltonian structures associated with involutive systems of ODEs. The proof of the correspondence is relied on the results of Sec. 4. In the final section, we briefly review the BRST formulation of the weak Hamiltonian structures in terms of two generating functions proposed in [6]. Then, using a superfield approach, we present a systematic procedure for explicit construction of the total BRST charge from the two generating functions of a weak Hamiltonian structure.

2. INVOLUTIVE SYSTEMS OF ODEs

In this paper, we consider autonomous systems of ordinary differential equations in the so-called *involutive normal form*:

$$\dot{x}^i + V^i(x) + \lambda^\alpha R_\alpha^i(x) = 0, \quad T_a(x) = 0. \quad (1)$$

Here the dot over x 's stands for the derivative in the independent variable t , called the “time”, and the dependent variables x 's and λ 's are treated as local coordinates on the phase space of

the system. To avoid topological complications we shall assume the phase space to be the linear manifold $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ with the global coordinates $\{x^i, \lambda^\alpha\}$. The vector field V entering the differential equations is called the *drift*, while the collection of the vector fields $R = \{R_\alpha\}$ is referred to as the *gauge distribution*. The algebraic equations defined by the functions $T_a(x)$, $a = 1, \dots, l$, are called the *constraints*. Involutivity implies that the following identities hold with some structure functions A, B, D, E and vector fields C, F :

$$\begin{aligned} [R_\alpha, T_a] &= A_{\alpha a}^b T_b, & [R_\alpha, R_\beta] &= B_{\alpha\beta}^\gamma R_\gamma - T_a C_{\alpha\beta}^a, \\ [V, T_a] &= D_a^b T_b, & [V, R_\alpha] &= E_\alpha^\beta R_\beta - T_a F_\alpha^a, \end{aligned} \tag{2}$$

Hereafter the square brackets denote the Schouten bracket in the space $\Lambda(\mathbb{R}^n) = \bigoplus_{k=0}^n \Lambda^k(\mathbb{R}^n)$ of smooth polyvector fields on \mathbb{R}^n . In particular, the bracket of a vector field v with a function f (0-vector field) is understood as the Lie derivative of the function along the vector field, $[v, f] = L_v f$ and the Schouten bracket of two vector fields is given by their commutator¹.

Let Σ denote the zero locus of the constraints, i.e., $\Sigma = \{x \in \mathbb{R}^n \mid T_a(x) = 0, a = 1, \dots, l\}$. In what follows we assume the variety Σ to be nonempty, the constraints T_a to be functionally independent and the vector fields R_α to be linearly independent at each point of Σ . This amounts to the full rank condition for appropriate matrices, namely,

$$\text{rank}(\partial_i T_a(x)) = l, \quad \text{rank}(R_\alpha^i(x)) = m, \quad \forall x \in \Sigma. \tag{3}$$

The first equality also ensures that $\Sigma \subset \mathbb{R}^n$ is a smooth submanifold. It is called the *constraint surface*.

The property of the system (1) “to be involutive”, being defined by (2), actually captures two different aspects, which should not be mixed up. To discuss either of them, let us first introduce the exterior ideal $I \subset \Lambda(\mathbb{R}^n)$ generated by the 1- and 0-vector fields V, R ’s, and T ’s that determine the system (1). Relations (2) mean that I is closed for the Schouten bracket, $[I, I] \subset I$, and hence I is a subalgebra of the graded Lie algebra $\Lambda(\mathbb{R}^n)$. From the geometrical viewpoint, this means that the gauge distribution R is tangent to and integrable on Σ and it remains to be so even when completed by the drift V . Furthermore, the restriction $R|_\Sigma$ of the gauge distribution to the constraint surface is invariant under the action of $V|_\Sigma$. The distribution $R|_\Sigma$, being integrable and having a constant rank, defines a regular foliation \mathcal{F} on Σ . The leaves of \mathcal{F} are called the *gauge orbits*. The space of leaves $M = \Sigma/\mathcal{F}$ is known as the *physical phase space*. Notice that $\dim M = n - l$ whenever M is a Hausdorff manifold. The terminology “gauge distribution” and

¹For the general definition of the Schouten bracket see formula (27) below.

“gauge orbits” is justified by the fact that the system (1) enjoys infinitesimal gauge symmetries of the form

$$\delta_\varepsilon x^i = -\varepsilon^\alpha R_\alpha^i(x), \quad \delta_\varepsilon \lambda^\alpha = \dot{\varepsilon}^\alpha + \varepsilon^\beta (E_\beta^\alpha(x) + \lambda^\gamma B_{\beta\gamma}^\alpha(x)), \quad (4)$$

ε ’s being infinitesimal gauge parameters. As a result, the system of equations (1) is underdetermined and one can choose λ ’s to be arbitrary functions of time. It is easy to see [9] that any gauge invariant t -local value associated with the phase space \mathbb{R}^{n+m} can be represented by a function on Σ which is constant along each gauge orbit. Therefore, the algebra of physical observables is isomorphic to the commutative algebra of functions on the physical phase space M . In the next sections, we shall rediscover and reinterpret the last fact within a cohomological analysis. The time evolution of the physical observables is generated by the drift V , or more precisely, by its projection on M . It is the involutivity of the distribution $R|_\Sigma$ that was the main reason in [9] to call the normal form (1) involutive.

Also, there is another reason to use the term “involutive”. It is related to a general notion of involution for the system of differential algebraic equations [19]. Loosely, a system is said to be involutive if it contains no implicit integrability conditions. For the systems of the form (1) these hidden integrability conditions may appear when one differentiates the constraints with respect to t and eliminates then the velocities \dot{x} with the help of the differential equations. In general, this can result in new algebraic constraints on the phase-space variables. Adding these new constraints to the original ones and extracting functionally independent among them, one can repeat the above procedure once and again producing further integrability conditions. This is known as the completion of a system to involution. Taking the total derivative of the constraints T_a and making use of relations (2), we find

$$\frac{dT_a}{dt} = \frac{\partial T_a}{\partial x^i} (\dot{x}^i + V^i + \lambda^\alpha R_\alpha^i) - (\lambda^\alpha A_{\alpha a}^b + D_a^b) T_b. \quad (5)$$

So, the time evolution preserves Σ and no new constraints on x ’s or λ ’s arise. In other words, the system (1) is involutive provided that the first and third conditions in (2) are satisfied. Notice that the absence of hidden integrability conditions implies simultaneously the presence of the *Noether identities* (5) among the equations of motion (1). Indeed, the total derivative of every algebraic equation (i.e., its differential consequence) has to be given by a linear combination of the algebraic and differential equations.

If the constraints $T_a(x)$ are chosen to be independent, then no other Noether identities can exist. Although the Lie closedness of the exterior ideal I ensures the involutivity of the system in the sense of the absence of integrability conditions, the converse is not true. In particular, the presence

of the Noether identities (5) has nothing to do with involutivity of the distribution $R|_{\Sigma}$. When the latter is not involutive, the system (1) is still underdetermined, but the corresponding gauge transformations involve higher derivatives of the gauge parameters ε^α , so that $\dim M < n - l$. The last situation is typical for the so-called affine control systems [20].

Due to the full rank conditions (3), both the gauge symmetry transformations (4) and the Noether identities (5) are *irreducible* in the usual sense [1].

In [9], it was shown that *any* system of ODEs can be locally brought to an involutive normal form (1),(2) by introducing auxiliary variables. The differential algebraic equations (1) with the structure functions subject to the involutivity conditions (2) can thus be taken as a starting-point for the general theoretical analysis of local dynamics governed by ODEs. Equations (1) can also be regarded as a generalization of the Dirac-Bergmann normal form known in the constrained Hamiltonian dynamics [1]. In the Hamiltonian situation, the algebraic equations $T_a = 0$ constitute the set of all the first and second class constraints (both primary and secondary), the variables λ^α correspond to the Lagrange multipliers to the first class constraints, whose Hamiltonian vector fields are identified with the generators R_α of the gauge distribution. Finally, the Poisson bracket of the Hamiltonian generates the drift V . Upon these identifications, the involutivity conditions (2) are equivalent to completeness of the set of Hamiltonian constraints.

The advantage of the involutive normal form over the other equivalent representations of ODEs is the simple structure of the gauge transformation (4) and the Noether identities (5), namely, the absence of higher derivatives. This will allow us to perform an exhaustive cohomological analysis of the system and give an explicit description for all the relevant groups of local BRST cohomology.

3. LOCAL BRST COMPLEX

Within the BRST formalism the equations of motion (1), the gauge transformations (4) and the Noether identities (5) are all incorporated in a single object Ω_1 called the *classical BRST charge*. The construction of Ω_1 is made by the homological perturbation theory and it works, in principle, for arbitrary systems of PDEs. Referring to [7] and [15] for details, here we just present the “cookbook recipe” for the system at hand. First, the space \mathbb{R}^{n+m} of the original variables x ’s and λ ’s is extended by the new variables η_i , η^a , c^α , and ξ_a usually called the *ghosts*. The number of η ’s, c ’s, and ξ ’s coincides, respectively, with the number of equations of motion, gauge symmetries, and Noether identities. It is convenient to introduce the collective notation $\varphi^I = \{x^i, \lambda^\alpha, \eta_i, \eta^a, c^\alpha, \xi_a\}$. At the next step the collection φ redoubles by adding the dual variables $\bar{\varphi}_J = \{\bar{x}_i, \bar{\lambda}_\alpha, \bar{\eta}^i, \bar{\eta}_a, \bar{c}_\alpha, \bar{\xi}^a\}$ called the momenta. The variable from the either collection are considered to be arbitrary functions

of time. Introducing the canonical Poisson bracket

$$\{\varphi^I(t), \varphi^J(t')\} = 0, \quad \{\bar{\varphi}_I(t), \varphi^J(t')\} = \delta_I^J \delta(t - t'), \quad \{\bar{\varphi}_I(t), \bar{\varphi}_J(t')\} = 0, \quad (6)$$

one can think of $\varphi^I(t)$ and $\bar{\varphi}_J(t)$ as coordinates on an infinite-dimensional phase-space V . For an obvious reason we shall call the points of V *trajectories*. The space V is actually a multigraded superspace. The gradings are defined by prescribing the following degrees to the dependent variables:

$$\text{gh}(x^i) = \text{gh}(\lambda^\alpha) = 0, \quad \text{gh}(\eta_i) = \text{gh}(\eta^a) = -1, \quad \text{gh}(c^\alpha) = 1, \quad \text{gh}(\xi_a) = -2,$$

$$\text{gh}(\bar{\varphi}_J) = -\text{gh}(\varphi^J),$$

$$\epsilon(\varphi^I) = \text{gh}(\varphi^I), \quad \epsilon(\bar{\varphi}_J) = \text{gh}(\bar{\varphi}_J) \pmod{2},$$

$$\deg(x^i) = \deg(\lambda^\alpha) = \deg(\bar{\eta}_i) = \deg(\bar{\eta}^a) = \deg(c^\alpha) = \deg(\bar{\xi}^a) = 0,$$

$$\deg(\eta^i) = \deg(\eta_a) = 1, \quad \deg(\bar{c}^\alpha) = \deg(\xi_a) = 2,$$

$$\text{Deg}(\varphi^I) = 0, \quad \text{Deg}(\bar{\varphi}_J) = 1.$$

The \mathbb{Z} -grading defined by the first and second lines is known as the *ghost number*. As is seen the ghost numbers of momenta are opposite to the ghost numbers of the “position coordinates”. Since we are dealing with a mechanical system without fermionic degrees of freedom, the Grassmann parity $\epsilon \in \mathbb{Z}_2$ of all the variables is uniquely determined by their ghost number (the third line). Besides, there are two auxiliary \mathbb{N} -gradings: the *resolution degree* and the *momentum degree* denoted respectively by \deg and Deg . The former is crucial for the homological perturbation theory (hence the name), while the latter just counts the number of momenta in homogenous expressions.

Let $\Phi^A = (\varphi^I, \bar{\varphi}_J)$ denote the whole set of coordinates on the infinite-dimensional phase-space of trajectories V . By a *local function* on V we mean a function $f(t)$ that depends on time through the trajectory $\Phi^A(t)$ and its t -derivatives up to some finite order, that is, $f = f(\Phi, \dot{\Phi}, \dots, \overset{(k)}{\Phi})$. The local functions form a graded supercommutative algebra with respect to the point-wise multiplication, which we denote by F . *Local functionals* on V are by definition integrals of local functions over closed intervals $I \subset \mathbb{R}$. In the sequel we shall assume the integration domain I to be fixed once and for all. Then each local functional $\int_I f dt$ is completely specified by its integrand, i.e., by the local function $f \in F$. The correspondence between local functionals and functions is not one-to-one. Indeed, if the boundary conditions for the admissible trajectories of V are chosen in such a way that $g|_{\partial I} = 0$ for some $g \in F$, then $\int_I (Dg) dt = 0$, where $D = d/dt$ is the operator

of total time derivative. To eliminate this ambiguity we impose an equivalence relation, whereby two local functionals are considered as equivalent if they only differ by boundary terms. In other words, the variational derivatives of equivalent functionals coincide. Then two local functions f and f' determine equivalent functionals iff $f' - f = Dg$ for some $g \in F$. This allows us to identify the equivalence classes of local functionals with the quotient F/DF of the space of local functions by the subspace of total time derivatives². Notice that the latter subspace is not a subalgebra in F .

The classical BRST charge Ω_1 is now defined to be a local functional satisfying the following set of conditions:

$$\begin{aligned}
(1) \quad & \epsilon(\Omega_1) = 1, \text{ gh}(\Omega_1) = 1, \text{ Deg}(\Omega_1) = 1; \\
(2) \quad & \Omega_1 = \int dt \left[\bar{\eta}_i (\dot{x}^i + V^i + \lambda^\alpha R_\alpha^i) + \bar{\eta}^a T_a \right. \\
& \quad + c^\alpha (\dot{\bar{\lambda}}_\alpha + R_\alpha^i \bar{x}_i + \bar{\lambda}_\beta E_\alpha^\beta + \lambda^\beta B_{\beta\alpha}^\gamma \bar{\lambda}_\gamma + \lambda^\beta C_{\alpha\beta}^{ai} \bar{\eta}_i \eta_a - F_\alpha^{ai} \bar{\eta}_i \eta_a - A_{\alpha a}^b \bar{\eta}^a \eta_b - \frac{\partial R_\alpha^i}{\partial x^j} \bar{\eta}_i \eta^j) \\
& \quad \left. - \bar{\xi}^a (\eta_b D_a^b + \lambda^\alpha A_{\alpha a}^b \eta_b - \eta^i \frac{\partial T_a}{\partial x^i} + \dot{\eta}_a) \right] + \dots,
\end{aligned} \tag{7}$$

where the dots stand for the terms at least quadratic in c 's and $\bar{\xi}$'s;

$$(3) \quad \{\Omega_1, \Omega_1\} = 0.$$

The first condition defines Ω_1 to be an odd functional of ghost number 1 with linear dependence of momenta. (The subscript 1 in the notation Ω_1 just points to the linear dependence of momenta.) The second condition defines the leading terms in the expansion of Ω_1 according to the resolution degree. The higher order terms are determined from the *classical master equation* (3) by means of the homological perturbation theory [1]. On this account one can regard (2) as a “boundary condition” for the master equation (3). Notice that the vanishing of the Poisson square of Ω_1 is a nontrivial condition to satisfy as the functional Ω_1 is odd. A general theorem proved in [15] ensures that the classical BRST charge always exists and is unique up to a canonical transformation in V .

The Hamiltonian vector field

$$s_0 = \{\Omega_1, \cdot\} = \delta + \gamma + \sum_{r=1}^{\infty} s_0^{(r)}, \tag{8}$$

²This equivalence relation on the space of local functionals is not so artificial as might appear at first sight. In actual fact it is customary to impose zero boundary conditions on all the ghosts and momenta as well as their derivatives. Then the integral of the total derivative of a local function with nonzero ghost number or momentum degree is equal to zero automatically. Another situation where the equivalence relation above establishes an isomorphism between the local functions and functionals is the case of differential equations on circle, $I = S^1$.

generated by the classical BRST charge is called the *classical BRST differential*. In the expression above it is expanded according to the resolution degree such that

$$\deg \delta = -1, \quad \deg \gamma = 0, \quad \deg s_0^{(r)} = r.$$

Clearly, the action of s_0 differentiates the algebra of local functions. The vector fields δ and γ are known as the *Koszul-Tate differential* and the *longitudinal differential*, respectively [1]. Since $s_0^2 = 0$ and $\text{gh } s_0 = 1$, the classical BRST differential makes the algebra F into a cochain complex with respect to the ghost number. Considering that $\text{Deg } s_0 = 0$, the complex F splits into the direct sum of complexes with definite momentum degree. We denote the corresponding cohomology groups by $H_m^g(s_0)$; here the superscript refers to the ghost number, while the subscript points on the momentum degree. Since the action of the variational vector field s_0 commutes with the time derivative, we have the short exact sequence of complexes

$$0 \longrightarrow DF \xrightarrow{i} F \xrightarrow{p} F/DF \longrightarrow 0$$

where i is the natural inclusion and p is the canonical projection. As we have explained above the quotient F/DF is naturally identified with the space of local functionals on V . Let us denote its cohomology groups by $H_m^g(s_0|D)$. We shall refer to $H_m^g(s_0|D)$ as the groups of *relative BRST cohomology* or cohomology of s_0 modulo D . The classes of relative BRST cohomology are given by the equivalence classes $f + s_0F + DF$ where $f \in F$ and $s_0f \in DF$.

The identity $s_0^2 = 0$, being expended with respect to the resolution degree, implies the infinite sequence of equalities

$$\delta^2 = 0, \quad [\delta, \gamma] = 0, \quad \gamma^2 = -[\delta, s_0^{(1)}], \quad \dots \quad (9)$$

As is seen, the Koszul-Tate differential squares to zero by itself defining thus one more coboundary operator in F and F/DF . Let us write $H_m^g(\delta)$ and $H_m^g(\delta|D)$ for the corresponding cohomology groups. Then the second and third relations in (9) suggest that the longitudinal differential γ induces a coboundary operator in the δ -cohomology: if $[f] \in H(\delta)$, then we set $\gamma([f]) = [\gamma f]$. (By abuse of notation we denote this induced coboundary operator by the same letter γ .) A similar definition applies to the relative δ -cohomology making the space $H(\delta|D)$ into a cochain complex with respect to γ . We let $H(\gamma, H(\delta))$ and $H(\gamma, H(\delta|D))$ denote the corresponding cohomology groups. Besides the momentum degree, these γ -cohomology groups are also graded by the resolution degree as $\deg \gamma = 0$.

Remark. Since the classical BRST charge Ω_1 is linear in momenta, the Hamiltonian action of s_0 is completely determined by its restriction on local functions with zero momentum degree. This

restriction defines a homological vector field on the φ -space, which is also called the classical BRST differential [7]. It is the terms of resolution degree -1 and 0 of this last homological vector field that are usually referred to as the Koszul-Tate and longitudinal differentials. Geometrically, one can think of s_0 as a canonical lift (the Lie derivative construction) of the homological vector field from the space of φ -trajectories to its cotangent bundle V .

Notice that the Koszul-Tate differential δ decreases the resolution degree exactly by one unit in contrast to s_0 , which is inhomogeneous. This allows us to interpret F and F/DF as the *chain* complexes with respect to the resolution degree. The corresponding homology groups in degree r will be denoted by $H^{(r)}(\delta) = \bigoplus H_m^{(r)}(\delta)$ and $H^{(r)}(\delta|D) = \bigoplus H_m^{(r)}(\delta|D)$. (We enclose the superscript in round brackets to distinguish it from the ghost number. The lower index indicates the momentum degree as before.) This change-over from the δ -cohomology to the δ -homology and vice versa is very helpful for formulating and proving various assertions below.

The local BRST cohomology of regular systems of PDEs was systematically studied in our recent paper [15]. Being applied to ODEs, the results of [15] lead to the conclusion that all the nontrivial BRST groups concentrate in resolution degrees 0 and 1 and are given by

$$\begin{aligned} H_m^g(s_0) &\cong H_m^g(\gamma, H^{(0)}(\delta)), & g \geq m \geq 0; \\ H_m^g(s_0|D) &\cong H_m^g(\gamma, H^{(0)}(\delta|D)), & g \geq m \geq 0; \\ H_{g+1}^g(s_0|D) &\cong H_{g+1}^{(1)}(\delta|D), & g \geq -1. \end{aligned}$$

From the viewpoint of physics, the most notable among these groups are the following:

- $H_0^0(s_0)$ the group of physical observables with values in local functions;
- $H_0^0(s_0|D)$ the group of physical observables with values in local functionals;
- $H_0^{-1}(s_0|D)$ the group of characteristics;
- $H_1^0(s_0|D)$ the group of rigid symmetries;
- $H_2^1(s_0|D)$ the group of Lagrange structures;
- $H_3^2(s_0|D)$ the group of potential obstructions to integrability of the Lagrange structures.

In the next section we study all these groups more closely.

4. THE LOCAL BRST COHOMOLOGY OF ODES

4.1. The group $H(\delta)$. The algebraic concept of *filtration* [21] considerably facilitates (or even makes possible) the computation of (co)homology groups. In our geometric setting, it comes from the natural filtration of the underlying jet space. Namely, let us arrange the variables

coordinatizing the vertical part of the infinite jet space $J^\infty V$ in the following increasing sequence of finite sets:

$$V_0 = \{x^i, \bar{\eta}_i, c^\alpha, \bar{\xi}^a, \eta_a, \bar{\lambda}_\alpha\},$$

$$V_s = V_{s-1} \cup \left\{ \overset{(s)}{x}^i, \overset{(s-1)}{\bar{x}}_i, \overset{(s-1)}{\eta}^i, \overset{(s)}{\bar{\eta}}_i, \overset{(s-1)}{\eta}^a, \overset{(s)}{\bar{\eta}}_a, \overset{(s-1)}{\lambda}^\alpha, \overset{(s)}{\bar{\lambda}}_\alpha, \overset{(s-1)}{\xi}^a, \overset{(s)}{\bar{\xi}}_a, \overset{(s)}{c}^\alpha, \overset{(s-1)}{\bar{c}}_\alpha \right\}, \quad s \in \mathbb{N}.$$

Associated to this sequence is the ascending filtration of the space of local functions

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_\infty = F, \quad F_s = C^\infty(V_s). \quad (10)$$

The filtration is chosen so as to be compatible with the action of the Koszul-Tate differential, that is, $\delta F_s \subset F_s$. The last property is easily seen from the following explicit expressions³:

$$\begin{aligned} \delta \overset{(k)}{\eta}^i &= \overset{(k+1)}{x}^i + D^k(V^i + \lambda^\alpha R_\alpha^i), & \delta \overset{(k)}{\xi}_a &= D^k \left(-\eta_b D_a^b - \lambda^\alpha A_{\alpha a}^b \eta_b + \eta^i \frac{\partial T_a}{\partial x^i} \right) - \overset{(k+1)}{\eta}_a. \\ \delta \overset{(k)}{\eta}_a &= D^k T_a, & \delta \overset{(k)}{\bar{\lambda}}_\alpha &= -D^k(\bar{\eta}_i R_\alpha^i), & \delta \overset{(k)}{\bar{x}}_i &= -D^k \left(\frac{\partial V^j}{\partial x^i} \bar{\eta}_j + \lambda^\alpha \frac{\partial R_\alpha^j}{\partial x^i} \bar{\eta}_j + \bar{\eta}^a \frac{\partial T_a}{\partial x^i} \right) + \overset{(k+1)}{\bar{\eta}}_i, \\ \delta \overset{(k)}{\bar{c}}_\alpha &= \overset{(k+1)}{\bar{\lambda}}_\alpha + D^k \left(R_\alpha^i \bar{x}_i + \bar{\lambda}_\beta E_\alpha^\beta + \lambda^\beta B_{\beta\alpha}^\gamma \bar{\lambda}_\gamma + \lambda^\beta C_{\alpha\beta}^{ia} \bar{\eta}_i \eta_a - F_\alpha^{ia} \bar{\eta}_i \eta_a - A_{\alpha a}^b \bar{\eta}^a \eta_b - \frac{\partial R_\alpha^i}{\partial x^j} \bar{\eta}_i \eta^j \right). \end{aligned}$$

(The other variables of V_s are annihilated by δ .) Here we introduced the Cartan vector field on $J^\infty V$,

$$D = \sum_{s=0}^{\infty} \overset{(s+1)}{\Phi}^A \frac{\partial}{\partial \overset{(s)}{\Phi}^A}, \quad (11)$$

which is nothing else but the jet counterpart of the operator of time derivative. It should be noted that the action of the Koszul-Tate differential defines (and is defined by) the boundary condition for the classical BRST differential [15]. So, no other terms than those written explicitly down in (7) are needed to find the action of δ in V_s .

The filtration (10) is exhaustive in the sense that each local function belongs to some F_s for s large enough. The natural inclusions $i_{ss'} : F_s \rightarrow F_{s'}$ for $s \leq s'$ induce the homomorphisms $i_{ss'}^* : H(F_s) \rightarrow H(F_{s'})$ of the homology groups associated with the direct system of complexes $\{F_s, i_{ss'}\}$ indexed by \mathbb{N} . As the homology functor commutes with direct limits⁴, we can define the δ -homology groups of the complex F by setting $H(\delta) = \varinjlim H(F_s)$. By definition of the direct limit any element of $H(\delta)$ is represented by a cycle that belongs to at least one space F_s .

³Notice that the most natural filtration of F with $F_k = C^\infty(J^k V)$ is not respected by δ .

⁴Even $\varinjlim F_s$ is just a union, $\{H(F_s), i_{ss'}^*\}$ is generally a nontrivial direct system as the homology functor does not preserve monomorphisms.

Since each complex F_s consists of smooth functions living on a *finite-dimensional* graded superdomain, we can freely apply to them all the usual differential-geometric constructions like the inverse function theorem. In particular, consider the change of variables V_s whereby

$$\begin{aligned} \overset{(k+1)}{x}_i &\mapsto \overset{k+1}{x}_i = \delta \overset{(k)}{\eta}^i, & \overset{(k+1)}{\eta}_a &\mapsto \overset{k+1}{\eta}_a = \delta \overset{(k)}{\xi}_a, \\ \overset{(k+1)}{\bar{\eta}}_i &\mapsto \overset{k+1}{\bar{\eta}}_i = \delta \overset{(k)}{\bar{x}}_i, & \overset{(k+1)}{\bar{\lambda}}_\alpha &\mapsto \overset{k+1}{\bar{\lambda}}_\alpha = \delta \overset{(k)}{\bar{c}}_\alpha, \end{aligned} \quad k = 0, 1, \dots, s-1,$$

and all the other variables of V_s remain the same. It is easy to see that this change of coordinate variables is nondegenerate and brings the Koszul-Tate differential to the form

$$\delta|_{F_s} = \sum_{k=0}^{s-1} \left(\overset{k+1}{x}_i \frac{\partial}{\partial \overset{(k)}{\eta}^i} + \overset{k+1}{\bar{\eta}}_i \frac{\partial}{\partial \overset{(k)}{\bar{x}}_i} + \overset{k+1}{\bar{\lambda}}_\alpha \frac{\partial}{\partial \overset{(k)}{\bar{c}}_\alpha} - \overset{k+1}{\eta}_a \frac{\partial}{\partial \overset{(k)}{\xi}_a} \right) - \bar{\eta}_i R_\alpha^i(x) \frac{\partial}{\partial \bar{\lambda}_\alpha} + T_a(x) \frac{\partial}{\partial \eta_a}.$$

(For $s = 0$ the first sum is absent.) Define the sequence of sets

$$V_0^0 = V_0, \quad V_s^0 = V_{s-1}^0 \cup \left\{ \overset{(s-1)}{\lambda}^\alpha, \overset{(s-1)}{\bar{\eta}}_a, \overset{(s)}{c}^\alpha, \overset{(s)}{\xi}_a \right\}, \quad s \in \mathbb{N}.$$

The complex F_s splits into the direct sum $F_s^0 \oplus F'_s$ of two subcomplexes, where the elements of F_s^0 are the smooth functions of the variables V_s^0 . Writing $\pi : F_s \rightarrow F_s^0$ for the natural projection that takes all the variables from $V_s \setminus V_s^0$ to zero, we can define the complementary subspace as $F'_s = (1 - \pi)F_s$. The invariance of F_s^0 and F'_s under the action of δ is obvious.

Now we claim that the complex F'_s is acyclic. Indeed, consider the operator

$$\sigma = \sum_{k=0}^{s-1} \left(\overset{(k)}{\eta}^i \frac{\partial}{\partial \overset{k+1}{x}_i} + \overset{(k)}{\bar{x}}_i \frac{\partial}{\partial \overset{k+1}{\bar{\eta}}_i} + \overset{(k)}{\bar{c}}_\alpha \frac{\partial}{\partial \overset{k+1}{\bar{\lambda}}_\alpha} - \overset{(k)}{\xi}_a \frac{\partial}{\partial \overset{k+1}{\eta}_a} \right),$$

which maps F'_s into itself and squares to zero. The anti-commutator of σ and δ is given by

$$N = \sum_{k=0}^{s-1} \left(\overset{k+1}{x}_i \frac{\partial}{\partial \overset{k+1}{x}_i} + \overset{k+1}{\bar{\eta}}_i \frac{\partial}{\partial \overset{k+1}{\bar{\eta}}_i} + \overset{k}{\eta}^i \frac{\partial}{\partial \overset{k}{\eta}^i} + \overset{k}{\bar{x}}_i \frac{\partial}{\partial \overset{k}{\bar{x}}_i} + \overset{k}{\bar{c}}_\alpha \frac{\partial}{\partial \overset{k}{\bar{c}}_\alpha} + \overset{k+1}{\bar{\lambda}}_\alpha \frac{\partial}{\partial \overset{k+1}{\bar{\lambda}}_\alpha} + \overset{k}{\xi}_a \frac{\partial}{\partial \overset{k}{\xi}_a} + \overset{k+1}{\eta}_a \frac{\partial}{\partial \overset{k+1}{\eta}_a} \right).$$

Since N is obviously invertible in F'_s , the composition $h = N^{-1}\sigma$ gives a contracting homotopy for $\delta|_{F'_s}$ and acyclicity of F'_s follows.

The restriction of the Koszul-Tate differential to F_s^0 is given by the operator

$$\delta|_{F_s^0} = T_a \frac{\partial}{\partial \eta_a} - \bar{\eta}_i R_\alpha^i \frac{\partial}{\partial \bar{\lambda}_\alpha},$$

which homology can be described as follows. Due to the irreducibility conditions (3) for the gauge symmetries and constraints, a function $a \in F_s^0$ is a δ -cycle iff it is independent of η_a 's and $\bar{\lambda}$'s. Let \tilde{F}_s^0 denoted the algebra of smooth functions of the variables $\tilde{V}_s^0 = V_s^0 \setminus \{\bar{\lambda}_\alpha, \eta_a\}$ and let I_s be

an ideal in \tilde{F}_s^0 generated by the functions $T_a(x)$ and $\bar{\eta}_i R_\alpha^i(x)$. It is clear that $I_s = \delta F_s^0 \cap \tilde{F}_s^0$, and hence

$$H(F_s) \cong H(F_s^0) \cong \tilde{F}_s^0 / I_s. \quad (12)$$

One can also give the group $H(F_s)$ a geometrical interpretation. The ideal I_s , being regular, defines a smooth submanifold M_s in the superdomain with coordinates \tilde{V}_s^0 . The “points” of M_s are solutions to the equations

$$T_a(x) = 0, \quad \bar{\eta}_i R_\alpha^i(x) = 0.$$

Then (12) says that the group $H(F_s)$ is isomorphic to the space of smooth functions on M_s .

Notice that all the elements of $H(F_s^0)$ have resolution degree zero. This is in agreement with the general property of the Koszul-Tate differential of being acyclic in positive resolution degree.

4.2. The group $H(\gamma, H(\delta))$. Having studied the δ -homology, we can now turn to the cohomology associated with the longitudinal differential. To do this requires an explicit expression for the action of γ on local functions. Unlike the Koszul-Tate differential, the Hamiltonian action of the boundary terms (7) by themselves do not specify the whole γ . In this case we need to know the classical BRST charge up to the first order in resolution degree. The missing terms of resolution degree 1 can easily be found from the classical master equation by means of the homological perturbation theory. Without going into detail we simply present the function that should be added to (7) to have Ω_1 specified up to the first order in resolution degree. It reads

$$\bar{c}_\alpha \Psi^\alpha + \xi_a \Theta^a + \bar{\lambda}_\alpha \tau \Psi^\alpha + \eta_a \tau \Theta^a, \quad (13)$$

where

$$\Psi^\alpha = \frac{1}{2} c^\gamma c^\beta B_{\beta\gamma}^\alpha, \quad \Theta^a = \frac{1}{2} c^\beta c^\gamma C_{\beta\gamma}^{ai} \bar{\eta}_i + c^\beta \bar{\xi}^b A_{\beta b}^a,$$

and we introduced the operator

$$\tau = \eta^i \frac{\partial}{\partial x^i} + \bar{x}_i \frac{\partial}{\partial \bar{\eta}_i}. \quad (14)$$

Notice that expression (13) (and hence, the first order BRST charge) involves only the structure functions of the involutivity conditions (2). Of course, higher orders in resolution degree, if any, involve new structure functions coming from the iterated commutators of V , R 's, and T 's. Important though these higher structure functions are for the definition of the classical BRST complex, they do not contribute to (the computation of) the classical BRST cohomology.

Summing (7) and (13) and extracting the zero-resolution-degree part in the classical BRST differential (8), we find

$$\begin{aligned}
\gamma \overset{(k)}{x}^i &= D^k (c^\alpha R_\alpha^i), & \gamma \overset{(k)}{\lambda}^\alpha &= D^k (c^\beta E_\beta^\alpha + c^\gamma \lambda^\beta B_{\beta\gamma}^\alpha) - \overset{(k+1)}{c}^\alpha, \\
\gamma \overset{(k)}{c}^\alpha &= D^k \Psi^\alpha, & \gamma \overset{(k)}{\eta}_i &= D^k \left(\bar{\xi}^a \frac{\partial T_a}{\partial x^i} - c^\alpha \frac{\partial R_\alpha^j}{\partial x^i} \bar{\eta}_j \right), & \gamma \overset{(k)}{\xi}^a &= -D^k \Theta^a, \\
\gamma \overset{(k)}{\eta}^a &= D^k (c^\alpha \lambda^\beta C_{\alpha\beta}^{ia} \bar{\eta}_i - c^\alpha F_\alpha^{ia} \bar{\eta}_i - c^\alpha A_{\alpha b}^a \bar{\eta}^b - \bar{\xi}^b D_b^a - \bar{\xi}^b \lambda^\alpha A_{\alpha b}^a) + \overset{(k+1)}{\xi}^a, \\
\gamma \overset{(k)}{\bar{x}}_i &= -D^k \left[c^\alpha \frac{\partial}{\partial x^i} (R_\alpha^j \bar{x}_j + \bar{\lambda}_\beta E_\alpha^\beta + \lambda^\beta B_{\beta\alpha}^\gamma \bar{\lambda}_\gamma + \lambda^\beta C_{\alpha\beta}^{aj} \bar{\eta}_j \eta_a - F_\alpha^{aj} \bar{\eta}_j \eta_a - A_{\alpha a}^b \bar{\eta}^a \eta_b \right. \\
&\quad \left. - \frac{\partial R_\alpha^m}{\partial x^j} \bar{\eta}_m \eta^j) - \bar{\xi}^a \frac{\partial}{\partial x^i} (\eta_b D_a^b + \lambda^\alpha A_{\alpha a}^b \eta_b - \eta^j \frac{\partial T_a}{\partial x^j}) \right], \\
\gamma \overset{(k)}{\lambda}_\alpha &= D^k (\bar{\xi}^a A_{\alpha a}^b \eta_b - c^\gamma \bar{\lambda}_\beta B_{\alpha\gamma}^\beta - c^\gamma C_{\gamma\alpha}^{ai} \bar{\eta}_i \eta_a), & \gamma \overset{(k)}{\xi}_a &= D^k \left(\xi_b c^\beta A_{\beta a}^b + \eta_b c^\beta \eta^i \frac{\partial A_{\beta a}^b}{\partial x^i} \right), \\
\gamma \overset{(k)}{c}_\alpha &= D^k \left(\bar{c}_\beta c^\gamma B_{\alpha\gamma}^\beta - \xi_a c^\gamma C_{\gamma\alpha}^{ai} \bar{\eta}_i + A_{\alpha b}^a \bar{\xi}^b \xi_a - \bar{\lambda}_\beta c^\gamma \eta^i \frac{\partial B_{\alpha\gamma}^\beta}{\partial x^i} + \eta_a c^\gamma \eta^j \frac{\partial C_{\gamma\alpha}^{ai}}{\partial x^j} \bar{\eta}_i \right. \\
&\quad \left. + \eta_a c^\gamma C_{\gamma\alpha}^{ai} \bar{x}_i - \eta_a \eta^i \frac{\partial A_{\alpha b}^a}{\partial x^i} \bar{\xi}^b \right), \\
\gamma \overset{(k)}{\eta}^i &= D^k \left(-c^\alpha \lambda^\beta C_{\alpha\beta}^{ai} \eta_a + c^\alpha F_\alpha^{ai} \eta_a + c^\alpha \frac{\partial R_\alpha^i}{\partial x^j} \eta^j \right), & \gamma \overset{(k)}{\eta}_a &= D^k (c^\alpha A_{\alpha a}^b \eta_b).
\end{aligned} \tag{15}$$

As is seen, γ respects the filtration (10) in the sense that $\gamma F_s \subset F_s$ and we can set $H(\gamma, H(\delta)) = \lim_{\rightarrow} H(\gamma, H(F_s))$.

Let us now show that the complex $(\gamma, H(\delta))$ is homotopic to its subcomplex $(\gamma, H(F_0^0))$ so that $H(\gamma, H(\delta)) \cong H(\gamma, H(F_0^0))$. To this end, introduce the operator

$$\sigma_s = \overset{(s-1)}{\eta}^a \frac{\partial}{\partial \overset{(s)}{\xi}^a} - \overset{(s-1)}{\lambda}^\alpha \frac{\partial}{\partial \overset{(s)}{c}^\alpha},$$

which maps F_s into itself. It is easy to see that σ_s anti-commutes with δ , inducing a well-defined operator in $H(F_s)$. Anti-commuting σ_s with γ , we get

$$N_s = [\sigma_s, \gamma] = \overset{(s)}{c}^\alpha \frac{\partial}{\partial \overset{(s)}{c}^\alpha} + \overset{(s-1)}{\lambda}^\alpha \frac{\partial}{\partial \overset{(s-1)}{\lambda}^\alpha} + \overset{(s)}{\xi}^a \frac{\partial}{\partial \overset{(s)}{\xi}^a} + \overset{(s-1)}{\eta}^a \frac{\partial}{\partial \overset{(s-1)}{\eta}^a}.$$

The nontrivial γ -cocycles are bound to center in the kernel of the operator N_s . Since $\text{Ker } N_s = F_{s-1}$, we infer that any γ -cocycle from $H(F_s)$ is cohomologous to one from $H(F_{s-1})$ and, by induction, $H(\gamma, H(F_s)) \cong H(\gamma, H(F_0))$. It remains to note that according to (12)

$$H(F_0) \cong H(F_0^0) \cong \tilde{F}_0^0 / I_0.$$

By definition, \tilde{F}_0^0 is the algebra constituted by the smooth functions of the variables $\tilde{V}_0^0 = \{x^i, \bar{\eta}_i, c^\alpha, \bar{\xi}^a\}$ and the ideal $I_0 \subset \tilde{F}_0^0$ is generated by the functions T_a and $\bar{\eta}_i R_\alpha^i$. It follows from (15) that both \tilde{F}_0^0 and I_0 are invariant under the action of the longitudinal differential and we can study the γ -cohomology directly in the quotient space \tilde{F}_0^0/I_0 . Letting $\gamma_0 = \gamma|_{\tilde{F}_0^0}$, we find

$$\gamma_0 = c^\alpha R_\alpha^i \frac{\partial}{\partial x^i} + \left(\bar{\xi}^a \frac{\partial T_a}{\partial x^i} - c^\alpha \bar{\eta}_j \frac{\partial R_\alpha^j}{\partial x^i} \right) \frac{\partial}{\partial \bar{\eta}_i} + \frac{1}{2} c^\gamma c^\beta B_{\beta\gamma}^\alpha \frac{\partial}{\partial c^\alpha} + \left(\frac{1}{2} c^\beta c^\gamma C_{\beta\gamma}^{ia} \bar{\eta}_i + c^\beta \bar{\xi}^b A_{\beta b}^a \right) \frac{\partial}{\partial \bar{\xi}^a}.$$

The inclusion $\gamma_0 I_0 \subset I_0$ follows immediately from the involutivity conditions (2). By definition, a class $a + I_0$ is a γ_0 -cocycle iff $\gamma_0 a \in I_0$; it is a γ_0 -coboundary iff $a = \gamma_0 b + c$ for some $b \in \tilde{F}_0^0$ and $c \in I_0$. This leads us to the identification

$$H(\gamma, H(\delta)) \cong \frac{\gamma_0^{-1} I_0}{\text{Im } \gamma_0 \cup I_0}. \quad (16)$$

Consider, for example, the γ -cohomology in ghost number zero. As the ghost numbers of the variables $\bar{\eta}_i$, c^α , and $\bar{\xi}^a$ are strictly positive, the representative cocycles are given by the functions of x 's considered modulo constraints T_a . A function $a(x)$ defines a γ -cocycle if

$$c^\alpha R_\alpha^i \frac{\partial a}{\partial x^i} = c^\alpha U_\alpha^a T_a \quad (17)$$

for some smooth functions $U_\alpha^a(x)$. In other words, the cocycle $a(x)$ is to be annihilated by the gauge distribution $R = \{R_\alpha\}$ on the constraint surface Σ , and two such cocycles are equivalent iff their difference vanishes on Σ . This is exactly the definition of the t -local physical observables we have discussed in Sec. 2. The physical role of the other groups, $H^g(\gamma, H^0(\delta))$ with $g > 0$, is not well understood.

4.3. The group $H(\delta|D)$. Now we proceed to the study of the relative δ -homology. As before, our main computational tool is the concept of filtration. A reliant filtration here is, of course, that induced by (10). The inclusions $i_{ss'} : F_s \rightarrow F_{s'}$ underlying the filtration (10) pass through the quotient F/DF giving rise to the direct system of complexes $\{F_s/DF_s, j_{ss'}\}$ with $j_{ss'}$ induced by $i_{ss'}$. This allows us to define the δ modulo D homology groups as the direct limit $H(\delta|D) = \lim_{\rightarrow} H(\delta, F_s/DF_{s-1})$. Let $D_s : F_s \rightarrow F_{s+1}$ denote the restriction of the operator (11) onto F_s . The operator D_s being a chain transformation, we have two short exact sequences of complexes

$$\begin{aligned} A : \quad 0 &\longrightarrow \text{Ker } D_s \xrightarrow{i_1} F_s \xrightarrow{p_1} \text{Im } D_s \longrightarrow 0, \\ B : \quad 0 &\longrightarrow \text{Im } D_s \xrightarrow{i_2} F_{s+1} \xrightarrow{p_2} \text{Coker } D_s \longrightarrow 0. \end{aligned} \quad (18)$$

With this notation the group $H(\delta|D)$ is given by the direct limit $\lim_{\rightarrow} H(\text{Coker } D_s)$.

There is a standard algebraic construction [21, p.46] that allows one to fit the induced map $D_s^* : H(F_s) \rightarrow H(F_{s+1})$ on homology into an exact sequence that relates the groups $H(F_s)$, $H(F_{s+1})$, and $H(\text{Coker} D_s)$. The construction goes as follows. First, one defines the *mapping cone* of the chain transformation D to be the chain complex $\text{Con} D_s = F_s \oplus F_{s+1}$ with differential

$$\delta(a, b) = (-\delta a, D_s a + \delta b), \quad \deg(a, b) = \deg b = \deg a + 1.$$

Since $\text{Ker} D = \mathbb{R}$ and $\text{Im } \delta \cap \text{Ker } D = 0$, we conclude that

$$H(\text{Con} D_s) \cong H(\text{Coker} D_s) \oplus \mathbb{R}, \quad (19)$$

where the second summand is generated by the 1-cycle $(1, 0)$. The natural injection $i : F_{s+1} \rightarrow \text{Con} D_s$ is a cochain transformation. The projection $p : \text{Con} D_s \rightarrow \bar{F}_s$ with $p(a, b) = a$ is also a chain transformation, if by \bar{F}_s we mean the complex F_s with the dimensions all lowered by 1 and differential $-\delta$. Thus we arrive at the short exact sequence of complexes

$$0 \longrightarrow F_{s+1} \xrightarrow{i} \text{Con} D_s \xrightarrow{p} \bar{F}_s \longrightarrow 0.$$

It is clear that $H(F_s) \cong H(\bar{F}_s)$ as vector spaces. The short exact sequence above gives rise to the triangle diagram

$$\begin{array}{ccc} H(F_s) & \xrightarrow{D_s^*} & H(F_{s+1}) \\ & \nwarrow p_* & \swarrow i_* \\ & H(\text{Con} D_s) & \end{array} \quad (20)$$

with exact vertices and D_s^* playing the role of the connecting homomorphism. Since we are dealing with complexes of vector spaces, the triangle diagram implies the existence of an isomorphism

$$H(\text{Con} D_s) \cong \text{Ker} D_s^* \oplus \text{Coker} D_s^*$$

and it remains to compute the kernel and cokernel of the operator D_s^* .

Remark. There is also another triangle diagram canonically associated to the mapping cone

$$\begin{array}{ccc} H(\text{Coker} D_s) & \xrightarrow{\partial} & H(\text{Ker} D_s) \\ & \nwarrow k_* & \swarrow j_* \\ & H(\text{Con} D_s) & \end{array} \quad (21)$$

This diagram is exact although the sequence of complexes

$$0 \longrightarrow \text{Ker} D_s \xrightarrow{j} \text{Con} D_s \xrightarrow{k} \text{Coker} D_s \longrightarrow 0$$

is not; here $ja = (i_1a, 0)$ for $a \in \text{Ker} D_s$, $k(a, b) = p_2b$, and $\partial = \partial_A \partial_B$ is the composition of connecting homomorphisms for (18). Since $\text{Ker} D_s = \mathbb{R} = H(\text{Ker} D_s)$, the group $H(\text{Con} D_s)$ splits as in (19).

Let us start with the space $\text{ker } D_s^*$. We know that $H(F_s) \cong \tilde{F}_s^0/I_s$, so that each class of δ -homology is represented by some function $f \in \tilde{F}_s^0$. Introducing the collective notation for the coordinates

$$z_0^A = (c^\alpha, \bar{\xi}^a), \quad z_s^A = \left(\binom{s-1}{\lambda} \alpha, \binom{s-1}{\bar{\eta}} a, \binom{s}{c} \alpha, \binom{s}{\bar{\xi}} a \right), \quad s \in \mathbb{N},$$

we can write the action of the operator D_s on $f \in \tilde{F}_s^0$ as

$$D_s f = D'_s f + Lf + \delta(\tau f), \quad (22)$$

where

$$D'_s = \sum_{k=0}^s z_{k+1}^A \frac{\partial}{\partial z_k^A}, \quad L = -V^i \frac{\partial}{\partial x^i} - \lambda^\alpha R_\alpha^i \frac{\partial}{\partial x^i} + \bar{\eta}_i \frac{\partial V^i}{\partial x^j} \frac{\partial}{\partial \bar{\eta}_j} + \lambda^\alpha \bar{\eta}_i \frac{\partial R_\alpha^i}{\partial x^j} \frac{\partial}{\partial \bar{\eta}_j} + \bar{\eta}^a \frac{\partial T_a}{\partial x^i} \frac{\partial}{\partial \bar{\eta}_i},$$

and the operator τ is defined by (14). Notice that the operator $D'_s + L$ maps the algebra \tilde{F}_s^0 into \tilde{F}_{s+1}^0 and the ideal I_s into I_{s+1} ; hence, its action descends to the δ -homology. Then the condition $[f] \in \text{Ker} D_s^*$ implies that

$$D'_s f + Lf = g \quad (23)$$

for some $g \in I_{s+1}$. Applying now the operator

$$\varrho_s = z_s^A \frac{\partial}{\partial z_{s+1}^A}$$

to both sides of equation (23), we get

$$z_s^A \frac{\partial f}{\partial z_s^A} = \varrho_s g$$

or, what is the same,

$$t \frac{d}{dt} f(tz_s^A) = (\varrho_s g)(tz_s^A).$$

This yields

$$f(z_s^A) - f(0) = \int_0^1 \frac{dt}{t} (\varrho_s g)(tz_s^A) \in I_s.$$

The function $\varrho_s g$ being proportional to z_s^A , the integral is well-defined and f appears to be homologous to some $f_0 \in \tilde{F}_{s-1}^0$. Proceeding in this way we see that f is cohomologous to a function with no dependence of z 's, i.e., to a function of the variables x^i and $\bar{\eta}_i$. Let \tilde{F}_{-1}^0 denote the space of such functions. For any $f \in \tilde{F}_{-1}^0$ equation (23) reduces to

$$Lf = g. \quad (24)$$

Since λ 's and $\bar{\eta}^a$'s enter the l.h.s. of (24) at most linearly, we can always take $g = \delta h$ with

$$h = \bar{\lambda}_\alpha(h^\alpha + \lambda^\beta h_\beta^\alpha + \bar{\eta}^a h_a^\alpha) + \eta_a(h^a + \lambda^\beta h_\beta^a + \bar{\eta}^b h_b^a)$$

and h 's being functions of x^i and $\bar{\eta}_i$. Denoting by L_0 the restriction of the operator L onto \tilde{F}_{-1}^0 we can summarize our consideration by the following compact formula

$$\text{Ker } D_s^* \cong L_0^{-1} I_1. \quad (25)$$

The structure of the last isomorphism is easily unfolded by reading formula (22) from right to left. It just says that if $f \in \tilde{F}_{-1}^0$ satisfies (24), that is, belongs to $L_0^{-1} I_1$, then the function $\tau f + h$ is a δ modulo D cycle. Since $\text{Ker } \tau|_{\tilde{F}_{-1}^0} = \mathbb{R}$, the relative cycle $\tau f + h$ is nontrivial whenever the function f is nonconstant. It is the space of constant functions that corresponds to the direct summand \mathbb{R} in (19). Thus, the assignment $f \mapsto \tau f + h$ defines an apimorphism

$$\nu : \tilde{F}_{-1}^0 \rightarrow H^{(1)}(\delta|D), \quad \text{Ker } \nu = \mathbb{R}. \quad (26)$$

By definition, the variables x^i , $\bar{\eta}_i$ are characterized by nonnegative ghost numbers and the same is true for the elements of the space \tilde{F}_{-1}^0 . The operator τ decreases the ghost number by one unit. Taking into account an obvious correlation between the ghost number and momentum degree as well as nilpotency of the odd variables $\bar{\eta}^i$, we can write

$$H^{(1)}(\delta|D) = \bigoplus_{g=-1}^{n-1} H_{g+1}^g(\delta|D).$$

To further clarify the isomorphism (25) let us consider a geometric interpretation of the space \tilde{F}_{-1}^0 . Namely, we can think of functions

$$a(x, \bar{\eta}) = \sum_{k=0}^{\infty} f(x)^{i_1 \dots i_k} \bar{\eta}_{i_1} \dots \bar{\eta}_{i_k} \in \tilde{F}_{-1}^0$$

as (inhomogeneous) polyvector fields on \mathbb{R}^n with odd variables $\bar{\eta}_i$ playing the role of the natural frame $\partial/\partial x^i$. In this terms the exterior product of two polyvector fields corresponds to the usual multiplication of functions, while the Schouten bracket passes to

$$[a, b] = \frac{\partial a}{\partial \bar{\eta}_i} \frac{\partial b}{\partial x^i} - (-1)^{(\epsilon(a)+1)(\epsilon(b)+1)} \frac{\partial b}{\partial \bar{\eta}_i} \frac{\partial a}{\partial x^i}. \quad (27)$$

Both the multiplication operations are known to be compatible in the sense of the graded Leibniz rule, so that we can speak of the Gerstenhaber (or odd Poisson) algebra of polyvector fields on \mathbb{R}^n . We denote this algebra by $\Lambda(\mathbb{R}^n) = \bigoplus \Lambda^p(\mathbb{R}^n)$. Now, introducing the exterior ideal $J \subset \Lambda(\mathbb{R}^n)$

generated by 0-vectors T_a and 1-vectors $R_\alpha^i \bar{\eta}_i$, we can reformulate the involutivity conditions (2) in the following way:

$$[J, J] \subset J, \quad [V, J] \subset J.$$

The first relation just says that the exterior ideal J is closed for the Schouten bracket and so defines an ideal of the Gerstenhaber algebra $\Lambda(\mathbb{R}^n)$. According to the second relation this ideal is invariant with respect to the drift vector field V . Define the V -invariant stabilizer of J in $\Lambda(\mathbb{R}^n)$ as

$$\Lambda^J(\mathbb{R}^n) = \{a \in \Lambda(\mathbb{R}^n) \mid [V, a] \subset J, [a, J] \subset J\}.$$

The space $\Lambda^J(\mathbb{R}^n)$ is clearly a subalgebra of $\Lambda(\mathbb{R}^n)$ containing J and we can introduce the quotient Gerstenhaber algebra $\Lambda_J(\mathbb{R}^n) = \Lambda^J(\mathbb{R}^n)/J$. Now the defining relation (24) for the relative δ -cocycle $\tau f + h$ implies that f represents an element of $\Lambda^J(\mathbb{R}^n)$,

$$[V, f] = T_a h^a + h^\alpha R_\alpha, \quad [T_a, f] = T_b h_a^b + h_a^\alpha R_\alpha, \quad [R_\alpha, f] = T_a h_\alpha^a + h_\alpha^\beta R_\beta.$$

This leads us to the following identification of the relative homology groups belonging to $\text{Ker} D_s^*$:

$$H_0^{(1)}(\delta|D) \cong \Lambda_J^0(\mathbb{R}^n)/\mathbb{R}, \quad H_m^{(1)}(\delta|D) \cong \Lambda_J^m(\mathbb{R}^n), \quad m = 1, \dots, n. \quad (28)$$

Notice that all these groups are nested in resolution degree 1. The general results on the local BRST cohomology obtained in [15], [17] suggest the following physical interpretation of the groups (28). The space $\Lambda_J^0(\mathbb{R}^n)$ is identified with the space of conservation laws, then the quotient⁵ $\Lambda_J^0(\mathbb{R}^n)/\mathbb{R}$ coincides with the space of characteristics of the system (1). The subalgebra $\Lambda_J^1(\mathbb{R}^n)$ is naturally identified with the Lie algebra of global symmetries. The space $\Lambda_J^2(\mathbb{R}^n)$ is isomorphic, by definition, to the space of nontrivial Lagrange structures. In more detail these Lagrange structures will be discussed in Sec. 6, where we shall identify them with the so-called weak Poisson brackets. Here, we only mention that a Lagrange structure $P \in \Lambda_J^2(\mathbb{R}^n)$ is called *integrable* if $[P, P] = 0 \in \Lambda_J^3(\mathbb{R}^n)$. This allows us to regard $\Lambda_J^3(\mathbb{R}^n)$ as the space of potential obstructions to integrability of Lagrange structures. In case $n = 2$, each Lagrange structure appears to be integrable for dimensional reasons. As for the groups (28) with momentum degree > 2 , their interpretation as “the spaces of” or “obstructions to” is obscure to us at present.

It remains to consider the cokernel of the operator D_s^* . Unfortunately, the description of the space $\text{Coker} D_s^*$ appears to be less explicit than the kernel space. We know that each element

⁵Any constant is obviously an integral of motion, but its gradient gives the zero characteristic.

$f \in \tilde{F}_{s+1}^0$ is a δ -cycle and thus a cycle of δ modulo D . From (22) it then follows that the relative cycles of the form $f = D'_s g + Lg$ span the image of D_s^* . Therefore

$$\text{Coker} D_s^* = \frac{\tilde{F}_{s+1}^0}{I_{s+1} \cup \text{Im}(L + D'_s)}, \quad H^{(0)}(\delta|D) \cong \lim_{\rightarrow} \text{Coker} D_s^*. \quad (29)$$

Notice that all the elements of $\text{Coker} D_s^*$ are nested in zero resolution degree.

To gain greater insight into what the groups (29) are about, consider a mechanical system without gauge symmetries and constraints, that is, a system of ordinary differential equations associated to the vector field V . Then the algebra $\tilde{F}_s^0 = \tilde{F}_{-1}^0$ can be identified with $\Lambda(\mathbb{R}^n)$, the ideal I_s is absent, and the operator $L + D'_s$ reduces to the commutator with the drift V . Therefore, $H^{(0)}(\delta|D) \cong \Lambda(\mathbb{R}^n)/[V, \Lambda(\mathbb{R}^n)]$. Now suppose V is a vector field that vanishes at $x_0 \in \mathbb{R}^n$ together with its first partial derivatives. Then the Schouten bracket $[V, W]$ vanishes at the point x_0 , too, for any polyvector field W . Therefore each polyvector field that does not vanish at x_0 represents a nontrivial class of the δ modulo D cohomology. This demonstrates nontriviality of the group $H^{(0)}(\delta|D)$ even for systems with trivial phase-space topology. On the other hand, if the vector field V can be rectified in the whole of \mathbb{R}^n (and so has no stationary points), then $H^{(0)}(\delta|D) = 0$. Indeed, in rectifying coordinates $V = \partial/\partial x^1$ and for any polyvector field W on \mathbb{R}^n we have the representation $W = [V, \tilde{W}]$ with $\tilde{W} = \int dx^1 W$.

4.4. The group $H(\gamma, H(\delta|D))$. We begin with a simple remark that in resolution degree zero any relative δ -cycle is necessarily an “absolute” one as there is no total derivatives of resolution degree minus one. This implies the isomorphism

$$H(\gamma, H(\delta|D)) \cong H(\gamma|D^*, H(\delta)), \quad (30)$$

using which we can set $H(\gamma, H(\delta|D)) = \lim_{\rightarrow} H(\gamma|D_s^*, H(F_s))$. Since all the δ -homology concentrates in resolution degree 0, so does the relative cohomology of γ , that is, $H(\gamma, H^{(1)}(\delta|D)) = 0$. Let γ_s denote the differential in $H(F_s)$ induced by the action of γ in F_s and let $H(\gamma_s) = \bigoplus H^g(\gamma_s)$ denote the corresponding cohomology group. Now to describe the relative γ -cohomology group (30) we can apply the mapping cone construction to the cochain transformations $D_s^* : H(F_s) \rightarrow H(F_{s+1})$ in perfect analogy to our computation of the relative δ -homology. This time, however, it is convenient to combine the “ γ -counterparts” of the exact triangle diagrams (20) and (21) into a

single diagram, which looks like:

$$\begin{array}{ccccc}
 H(\gamma, H(F_0^0)) \cong H(\gamma_s) & \xrightarrow{\tilde{D}_s} & H(\gamma_{s+1}) \cong H(\gamma, H(F_0^0)) & & (31) \\
 & \nwarrow \beta & \swarrow & & \\
 & H(\text{Con}D_s^*) & & & \\
 & \swarrow & \nwarrow \alpha & & \\
 H(\text{Coker}D_s^*) & \xrightarrow{\quad} & H(\text{Ker}D_s^*) & &
 \end{array}$$

Here \tilde{D}_s is the operator induced by D_s^* on cohomology. The group we are interested in is given by the lower left corner of the diagram. In principle, it can be computed from the bottom triangle provided we know the groups $H(\text{Ker}D_s^*)$ and $H(\text{Con}D_s^*)$. By definition, the group $\text{Con}(D_s^*)$ is given by the direct sum $H(F_s) \oplus H(F_{s+1})$ endowed with the action of the coboundary operator γ :

$$\gamma(a, b) = (-\gamma_s a, D_s^* a + \gamma_{s+1} b), \quad \text{gh}(a, b) = \text{gh } b = \text{gh } a - 1.$$

The homomorphisms α and β are induced by the natural imbedding $a \mapsto (a, 0)$ and the natural projection $(a, b) \mapsto a$. It follows from the top triangle in (31) that

$$H(\text{Con}D_s^*) \cong \text{Ker}\tilde{D}_s \oplus \text{Coker}\tilde{D}_s.$$

To compute the kernel and cokernel of the operator \tilde{D}_s consider the identity

$$Df = Kf + \delta(\tau f) + [\gamma, \rho]f, \quad (32)$$

which holds for any $f \in F_0^0$; here the operators K and ρ are give by

$$K = -V^i \frac{\partial}{\partial x^i} + \bar{\eta}_j \frac{\partial V^j}{\partial x^i} \frac{\partial}{\partial \bar{\eta}_i} + c^\beta E_\beta^\alpha \frac{\partial}{\partial c^\alpha} + (c^\alpha F_\alpha^{ia} \bar{\eta}_i + \bar{\xi}^b D_b^a) \frac{\partial}{\partial \bar{\xi}^a}, \quad \rho = \bar{\eta}^a \frac{\partial}{\partial \bar{\xi}^a} - \lambda^\alpha \frac{\partial}{\partial c^\alpha}.$$

Since the operator ρ anti-commutes with δ , we can interpret equality (32) by saying that when restricted to γ -cocycles from \tilde{F}_0^0 the action of D coincides with that of K modulo γ -coboundaries. Furthermore, as one can easily verify, the operator K leaves invariant the spaces \tilde{F}_0^0 , I_0 and $I_0 \cup \text{Im}\gamma_0$, inducing thus a well-defined operator \tilde{K} in the quotient space $\gamma_0^{-1}I_0/(I_0 \cup \text{Im}\gamma_0)$. By virtue of the isomorphism (16) we can identify the spaces $\text{Ker}\tilde{D}_s$ and $\text{Ker}\tilde{K}$. Explicitly,

$$\text{Ker}\tilde{D}_s \cong \text{Ker}\tilde{K} \cong \frac{\gamma_0^{-1}I_0 \cap K^{-1}(I_0 \cup \text{Im}\gamma_0)}{I_0 \cup \text{Im}\gamma_0}.$$

For the kernel of \tilde{D}_s we have the following representation:

$$\text{Coker}\tilde{D}_s \cong \text{Coker}\tilde{K} \cong \frac{\gamma_0^{-1}I_0}{I_0 \cup \text{Im}\gamma_0 \cup K\gamma_0^{-1}I_0}.$$

With this result on the cohomology of the mapping cone, we can now turn to the study of the group $H(\text{Coker}D_s^*)$ entering the bottom triangle of diagram (31). There are two observations about this exact triangle: (i) the operator γ induces the zero differential in $\text{Ker}D_s^*$, so that $H(\text{Ker}D_s^*) \cong \text{Ker}D_s^*$, and (ii) the homomorphism α is an injection. The first fact is easily seen by comparing the action of the operators L_0 and γ_0 in \tilde{F}_{-1}^0 , while the second assertion follows immediately from injectivity of the composition $\beta\alpha$. The details are left to the reader. The homomorphism α being injective, the bottom triangle in (31) reduces to the short exact sequence

$$0 \longrightarrow \text{Ker}D_s^* \longrightarrow \text{Ker}\tilde{D}_s \oplus \text{Coker}\tilde{D}_s \longrightarrow H(\text{Coker}D_s^*) \longrightarrow 0 ,$$

where we made use of the established isomorphisms. Taking into account that $\text{Im}\alpha \subset \text{Ker}\tilde{D}_s$, we can write

$$H(\text{Coker}D_s^*) \cong \frac{\text{Ker}\tilde{D}_s}{\text{Ker}D_s^*} \bigoplus \text{Coker}\tilde{D}_s . \quad (33)$$

The quotient in the right hand side can be understood as follows. By (32), a γ -cocycle $f \in \tilde{F}_0^0$ gives rise to an element from $\text{Ker}\tilde{D}_s$ iff $Kf = \gamma h + g$ for some $h \in \tilde{F}_0^0$ and $g \in I_0$. Read from right to left Eq. (32) says that any such f defines the relative γ -cocycle $\rho f + h$. Among the elements of $\text{Ker}\tilde{D}_s$ are the “absolute” γ -cocycles from \tilde{F}_{-1}^0 . Being independent of c ’s and $\bar{\xi}$ ’s, these γ -cocycles are all annihilated by the operator ρ and so do not contribute to the relative γ -cohomology. This explains the structure of the first summand in (33).

The above consideration can now be summarized in the following formula:

$$H(\gamma, H(\delta|D)) \cong \frac{\gamma_0^{-1}I_0 \cap K^{-1}(I_0 \cup \text{Im}\gamma_0)}{I_0 \cup \text{Im}\gamma_0 \cup (\gamma_0^{-1}I_0 \cap \tilde{F}_{-1}^0)} \bigoplus \frac{\gamma_0^{-1}I_0}{I_0 \cup \text{Im}\gamma_0 \cup K\gamma_0^{-1}I_0} .$$

We close this section with an explicit example clarifying the geometric origin of the (relative) γ -cohomology. Consider the following system of differential algebraic equations:

$$\dot{x} - \lambda y = 0 , \quad \dot{y} + \lambda x = 0 , \quad x^2 + y^2 - 1 = 0 . \quad (34)$$

Comparing these equations with the general form of an involutive system (1), it is easy to see that the constraint surface Σ is given here by the unit circle standardly imbedded in xy -plane and the gauge distribution is spanned by the single vector field $R = x\partial_y - y\partial_x$ generating rotations. One can also check that the system meets both the involutivity (2) and full rank (3) conditions. By (4) and (5) each variable λ results in a gauge symmetry, and each constraint gives rise to a Noether identity. In the case at hand, the gauge transformations are given by

$$\delta_\varepsilon x = \varepsilon y , \quad \delta_\varepsilon y = -\varepsilon x , \quad \delta_\varepsilon \lambda = \dot{\varepsilon} , \quad (35)$$

while the Noether identity has the form

$$2x(\dot{x} - \lambda y) + 2y(\dot{y} + \lambda x) - D(x^2 + y^2 - 1) = 0.$$

The gauge orbits foliate the plane onto concentric circles, one of which coincides with the constraint surface. The physical phase space, being isomorphic to the quotient Σ/\sim , is given by a point, so that the system possesses no physical degrees of freedom. As a result, the space of local physical observables is exhausted by constant functions, $H^0(s_0) \cong H^0(\gamma, H(\delta)) \cong \mathbb{R}$. However, as we shall see in a moment, there are nontrivial physical observables with values in local functionals. Whereas the constants are physically observable by definition (the ground field), the presence of nonlocal observables is not a common property shared by all dynamical systems. The classical BRST charge associated to our system reads

$$\begin{aligned} \Omega_1 = \int dt \Big\{ & \bar{\eta}_x(\dot{x} - \lambda y) + \bar{\eta}_y(\dot{y} + \lambda x) + \bar{\eta}(x^2 + y^2 - 1) \\ & + c(y\bar{x} - x\bar{y} + \eta_y\bar{\eta}_x - \eta_x\bar{\eta}_y - \dot{\bar{\lambda}}) + \bar{\xi}(x\eta_x + y\eta_y - \tfrac{1}{2}\dot{\eta}) \Big\} \end{aligned}$$

As there are no higher structure functions, the classical BRST differential is mere the sum of the Koszul-Tate and longitudinal differentials,

$$s_0 = \delta + \gamma, \quad s_0^2 = \delta^2 = \gamma^2 = 0.$$

Now one can easily see that the gauge ghost c is BRST invariant,

$$\delta c = 0, \quad \gamma c = 0 \quad \Rightarrow \quad s_0 c = 0,$$

and the corresponding classes of s_0 - and γ -cohomology are nontrivial. Indeed, if they were trivial, there would exist a smooth function f of x and y such that

$$c = \gamma f + \delta g \tag{36}$$

for some g . Let us introduce the polar coordinate system (r, φ) instead of the Cartesian coordinates (x, y) . Then in a collar neighborhood of the constraint surface $r = 1$, the function f can be regarded as a smooth function of r and φ such that $f(r, \varphi + 2\pi) = f(r, \varphi)$. The generator of the gauge distribution takes the form $R = \partial_\varphi$. Equation (36) implies that

$$(\partial_\varphi f)(1, \varphi) = 1,$$

whatever the function g . But the last equality is impossible as the derivative of a periodic function must vanish at least at two points.

Applying the operator ρ yields a nontrivial class of the relative γ -cohomology, namely, $\lambda = -\rho c$. The BRST invariance of the integral $\Lambda = \int \lambda dt$ amounts to its gauge invariance provided that the gauge parameter obeys the zero boundary conditions:

$$s_0\Lambda = \int \dot{c}dt = 0 \quad \Leftrightarrow \quad \delta_\varepsilon\Lambda = \int \dot{\varepsilon}dt = 0.$$

The gauge invariance of the functional Λ admits also a purely geometric explanation. Let us treat x and y as 0-forms and λ as a 1-form on the time interval. Using the equations of motion (34), we can bring the gauge transformations (35) to the form of infinitesimal reparametrizations:

$$\delta_\varepsilon x = \dot{x}\tilde{\varepsilon}, \quad \delta_\varepsilon y = \dot{y}\tilde{\varepsilon}, \quad \delta_\varepsilon\lambda = D(\tilde{\varepsilon}\lambda), \quad \varepsilon = \tilde{\varepsilon}\lambda.$$

Then the functional Λ is given by the integral of the 1-form λ over an interval (a one-dimensional manifold with boundary), and hence it is invariant under diffeomorphisms. Thus, we are lead to conclude that not only do the BRST cohomology groups carry some valuable information about the physical sector of the theory, but they also ‘feel’ a particular realization of the physical phase space by means of imbedding and/or factorization.

5. THE TOTAL BRST CHARGE

The classical BRST charge, as its name suggests, incorporates all the ingredients of the classical theory: the equations of motion, their gauge symmetries and Noether identities. The corresponding BRST complex provides concise and rigorous definitions for such important notions of classical dynamics as physical observables, rigid symmetries, and conservation laws. Whereas the classical equations of motion are enough to formulate the classical dynamics they are certainly insufficient for constructing a quantum-mechanical description of the system. Any quantization procedure has to involve one or another additional geometric/algebraic structure. Within the path-integral quantization, for instance, it is the action functional that plays the role of such an additional structure. The procedure of canonical quantization relies on the Hamiltonian form of dynamics, involving a non-degenerate Poisson bracket and a Hamiltonian. Either approach assumes the existence of a variational formulation for the classical equations of motion (the least action principle), and becomes inapplicable beyond the scope of variational dynamics. The extension of these quantization methods to general non-variational systems was proposed in [6], [7]. In both the cases the structure responsible for quantization is obtained as the deformation of the corresponding classical BRST differential in the category of L_∞ -algebras; in so doing, the classical BRST differential is identified with the first structure map L_1 . For the most part, the quantum properties of the theory are determined by the second structure map L_2 , that is, the first order deformation of the

classical BRST differential. In the Hamiltonian picture of dynamics, or still better the phase-space approach, L_2 is identified with a weak Poisson structure [6], while in the Lagrangian or covariant approach it is known as a Lagrange structure [7]. It goes without saying that different choices for the deformation of classical BRST differential can generally result in different quantum theories.

The aim of this and the next two sections is to explain a relationship between the two mentioned approaches to quantization of non-variational gauge systems in the case of mechanical systems brought to the involutive normal form (1). To begin with we recall the definition of the total BRST charge.

Just as the path-integral quantization of Lagrangian gauge theories is formulated by means of a master action on the ghost-extended configuration space of fields, so the covariant quantization of non-variational theories is defined in terms of a single functional called the BRST charge. The latter can be viewed as the deformation of the classical BRST charge Ω_1 by terms of higher momentum degree,

$$\Omega = \Omega_1 + \sum_{p=2}^{\infty} \Omega_p, \quad \text{Deg } \Omega_p = p. \quad (37)$$

The only condition on the deformation (besides being local, Grassmann odd, and of ghost number 1) is that the total BRST charge Ω obeys the same master equation as the classical one, i.e.,

$$\{\Omega, \Omega\} = 0. \quad (38)$$

On substituting the expansion (37) into (38), we get the infinite chain of equations

$$\{\Omega_1, \Omega_1\} = 0, \quad \{\Omega_1, \Omega_2\} = 0, \quad \{\Omega_2, \Omega_2\} = 2\{\Omega_1, \Omega_3\}, \quad \dots \quad (39)$$

The first equation is automatically satisfied for the classical BRST charge Ω_1 . Then the second equation identifies the leading term of the deformation, Ω_2 , as a relative cocycle of the classical BRST differential $s_0 = \{\Omega_1, \cdot\}$. The deformation is called *regular* if $[\Omega_2] \neq 0 \in H_2^1(s_0|D)$ and *trivial* if Ω is canonically equivalent to Ω_1 . In the latter case there exists an even local functional G of ghost number zero such that

$$\Omega = e^{\{G, \cdot\}} \Omega_1, \quad \text{Deg } G \geq 2.$$

As the canonically equivalent systems are physically indistinguishable, we can confine ourselves to considering only nontrivial deformations. In our previous paper [15] the following alternative was proven: every deformation of the classical BRST charge associated to a mechanical system is either regular or trivial. Non-triviality of the class $[\Omega_2]$ is, of course, only a necessary condition for the existence of a nontrivial deformation starting with Ω_2 . As is usual in deformation theory, the necessary and sufficient condition for the existence of a regular deformation is that all the Massey

powers of $[\Omega_2]$ can be made zero simultaneously [15]. Indeed, due to the Jacoby identity the Poisson square of the cocycle Ω_2 is annihilated by s_0 , and hence, we have the class $[\{\Omega_2, \Omega_2\}] \in H_3^2(s_0|D)$. This class, denoted usually by $\langle[\Omega_2], [\Omega_2]\rangle$, is known as the Massey square of $[\Omega_2]$. One can easily see that the Massey square depends actually on the class $[\Omega_2]$, and not on its particular representative Ω_2 . For a general discussion of the Mossey products in the category of graded Lie algebras we refer the reader to [22], [23]. The Poisson bracket $\{\Omega_2, \Omega_2\}$, representing the Massey square, enters the left hand side of the third equation in (39). Since the the right hand side of the equation is proportional to the coboundary $s_0\Omega_3$, we are lead to conclude that the second order deformation Ω_2 of the classical BRST charge extends to the third order iff $\langle[\Omega_2], [\Omega_2]\rangle = 0$. Actually, it is also the necessary and sufficient condition for the existence of the total BRST charge Ω . The reason is that all the higher Massey powers

$$\langle \underbrace{[\Omega_2], [\Omega_2], \dots, [\Omega_2]}_m \rangle, \quad m = 3, 4, \dots,$$

belong to the groups $H_{m+1}^2(s_0|D)$ which are known to vanish for mechanical systems [15].

The total BRST charge admits also an interesting algebraic interpretation, which gives a further elucidating glimpse into the nature of regular deformations and their relation to the basic ingredients of the Batalin-Vilkovisky (BV) formalism [1]. Let \mathcal{A} denote the space of local functionals of momentum degree zero. In [7], it was observed that each total BRST charge (37) endows the space \mathcal{A} with the structure of L_∞ -algebra [24]. The corresponding structure maps $L_n : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}$ are defined through the derived bracket construction [25]:

$$L_n : a_1 \otimes a_2 \otimes \dots \otimes a_n \mapsto (a_1, a_2, \dots, a_n) = \{\dots \{\Omega_n, a_1\}, a_2\}, \dots, a_n\} \in \mathcal{A}. \quad (40)$$

In particular, the first structure map is given simply by the classical BRST differential s_0 and the second structure map defines the 2-bracket

$$(a, b) = \{\{\Omega_2, a\}, b\} \quad \forall a, b \in \mathcal{A}. \quad (41)$$

By definition, the 2-bracket is Grassmann odd and graded symmetric,

$$(a, b) = (-1)^{\epsilon(a)\epsilon(b)}(b, a) \quad \forall a, b \in \mathcal{A}.$$

As for the graded Jacobi identity, it is replaced by the following relation:

$$((a, b), c) + (-1)^{\epsilon(b)\epsilon(c)}((a, c), b) + (-1)^{\epsilon(a)(\epsilon(b)+\epsilon(c))}((a, b), c) = -\Delta(a, b, c), \quad (42)$$

where the trilinear functional Δ , describing deviation from the standard Jacobi identity, is determined by the third order term in the total BRST charge (37), namely,

$$\Delta(a, b, c) = s_0(a, b, c) + (s_0 a, b, c) + (-1)^{\epsilon(a)\epsilon(b)}(a, s_0 b, c) + (-1)^{(\epsilon(a)+\epsilon(b))\epsilon(c)}(a, b, s_0 c).$$

Following the physical terminology, we call (41) the *weak anti-bracket* and refer to (42) as the *weak Jacobi identity*. The second relation in (39) implies that the classical BRST differential s_0 and the weak anti-bracket are compatible in the sense of the graded Leibniz rule

$$s_0(a, b) = -(s_0 a, b) - (-1)^{\epsilon(a)}(a, s_0 b) \quad \forall a, b \in \mathcal{A}. \quad (43)$$

As a consequence, the weak anti-bracket descends to the classical BRST cohomology, inducing an odd Lie bracket in the space $H_0(s_0)$.

In a particular case, where the expansion (37) for the total BRST charge stops at the second term, i.e., $\Omega = \Omega_1 + \Omega_2$, the bracket (41) enjoys all the properties of the usual BV anti-bracket [1], including the Jacobi identity. If we further assume the anti-bracket to be non-degenerate, then the classical BRST differential is necessarily given by an anti-Hamiltonian vector field $s_0 = (S, \cdot)$ generated by some BV master action $S \in \mathcal{A}$. The latter obeys the classical master equation $(S, S) = 0$ by virtue of $s_0^2 = 0$. This is the most concise, though a somewhat formal, explanation of how the standard BV formalism for Lagrangian systems fits in this more general quantization approach.

6. THE LAGRANGE STRUCTURE AND THE WEAK HAMILTONIAN STRUCTURE

The discussion of the previous section can be summarized by saying that the total BRST charge Ω of a mechanical system is completely specified (up to canonical transform) by a classical BRST charge Ω_1 and a relative BRST cocycle Ω_2 satisfying the only condition

$$\langle [\Omega_2], [\Omega_2] \rangle = 0. \quad (44)$$

It is the condition which ensures that the weak anti-bracket (41) in \mathcal{A} induces a genuine anti-bracket (=odd Lie bracket) in the cohomology space $H_0(s_0|D)$.

We are now going to examine equation (44) more closely, using our knowledge about the structure of the local BRST cohomology associated to involutive systems of ODEs. Investigation of this question will lead us eventually to establishing an explicit one-to-one correspondence between the concepts of a Lagrange structure [7] and a weak Hamiltonian structure [6] for this particular class of dynamical systems.

In Section 4.3, we have shown the existence of the short exact sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{\mu} \Lambda_J(\mathbb{R}^n) \xrightarrow{\nu} H^{(1)}(\delta|D) \longrightarrow 0 ,$$

where the monomorphism μ is the natural inclusion and the epimorphism ν is defined by Eq. (26). The space $\Lambda_J(\mathbb{R}^n)$ carries the structure of a graded Lie algebra with the Lie bracket induced by the Schouten bracket on polyvector fields. Notice that the space $\text{Im } \mu \cong \mathbb{R}$, being identified with the space of constant functions on \mathbb{R}^n , belongs to the center of $\Lambda_J(\mathbb{R}^n)$. This allows us to define the quotient Lie algebra $\Lambda_J(\mathbb{R}^n)/\mathbb{R}$, whose carrier vector space is, by definition, isomorphic to $H^{(1)}(\delta|D)$. The push forward of the Lie bracket on $\Lambda_J(\mathbb{R}^n)/\mathbb{R}$ by means of ν defines then the Lie algebra structure on the cohomology space $H^{(1)}(\delta|D)$. Namely, if a and b are two elements of $H^{(1)}(\delta|D)$ such that $a = \nu(\alpha)$ and $b = \nu(\beta)$ for some $\alpha, \beta \in \Lambda_J(\mathbb{R}^n)$, then

$$\{a, b\} = \nu([\alpha, \beta]) . \quad (45)$$

Here we deliberately denote the push forward Lie bracket on $H^{(1)}(\delta|D)$ by braces. The reason is that the right hand side of (45) exactly coincides with the cohomology class of the Poisson bracket of relative δ -cocycles representing the classes a and b . The last fact can also be seen from the following construction. As established in [15], the group $H^{(1)}(\delta|D)$ is isomorphic to the direct product $\Pi = \bigoplus_{g=-1}^{\infty} H_{g+1}^g(s_0|D)$. The corresponding isomorphism $\varkappa : H^{(1)}(\delta|D) \rightarrow \Pi$ is defined in the following way. Each representative cocycle a of a class $[a] \in \Pi$ can be expanded according to the resolution degree,

$$a = a^{(1)} + a^{(2)} + a^{(3)} + \cdots , \quad \deg a^{(r)} = r .$$

The leading term has resolution degree 1 and is annihilated by the Koszul-Tate differential. By definition, we set $\varkappa([a]) = [a^{(1)}] \in H^{(1)}(\delta|D)$. Since the action of the classical BRST differential s_0 is Hamiltonian, the Poisson bracket on the space of local functionals passes through the cohomology making Π into a graded Lie algebra. The pull back of this Lie algebra structure via the isomorphism \varkappa gives the above Lie bracket (45) on $H^{(1)}(\delta|D)$. Thus, we arrive at the following commutative diagram of the Lie algebra isomorphisms:

$$\begin{array}{ccc} & H^{(1)}(\delta|D) & \\ \nu \nearrow & & \searrow \varkappa \\ \Lambda_J(\mathbb{R}^n)/\mathbb{R} & \xrightarrow{\varkappa\nu} & \Pi = \bigoplus_{g=-1}^{\infty} H_{g+1}^g(s_0|D) \end{array} \quad (46)$$

Let us now come back to the regular deformation (37) governed by the class $[\Omega_2] \in H_2^1(s_0|D)$. In view of the comments above this class has uniquely defined preimages in $\Lambda_J^2(\mathbb{R}^n)$ and $H_2^{(1)}(\delta|D)$:

$$[\Omega_2] = \varkappa([L]) = \varkappa\nu([P]). \quad (47)$$

The element $[L]$ of $H_2^{(1)}(\delta|D)$ is known as the *Lagrange structure* [15]. We see that for the involutive systems of ODEs, each Lagrange structure defines (and is defined by) a unique class $[P]$ of $\Lambda_J^2(\mathbb{R}^n)$. By the definition of $\Lambda_J^2(\mathbb{R}^n)$, the bivector field $P = P^{ij}\partial_i \wedge \partial_j$, representing the class $[P]$, obeys the relations

$$[T_a, P] = -Y_a^\alpha R_\alpha - T_b G_a^b, \quad [R_\alpha, P] = W_\alpha^\beta \wedge R_\beta - T_a M_\alpha^a, \quad [V, P] = Z^\alpha \wedge R_\alpha - T_a N^a \quad (48)$$

for some polyvector fields Y, G, W, M, Z, N . Applying the map (26), one can see that the corresponding Lagrange structure $[L] = \nu([P])$ is represented by the relative δ -cocycle

$$\begin{aligned} L = & 2P^{ij}\bar{x}_i\bar{\eta}_j + \eta^k\partial_k P^{ij}\bar{\eta}_i\bar{\eta}_j \\ & - Z^{\alpha i}\bar{\lambda}_\alpha\bar{\eta}_i + \eta_a N^{aij}\bar{\eta}_i\bar{\eta}_j + Y_a^\alpha\bar{\lambda}_\alpha\bar{\eta}^a + \eta_b G_a^{bi}\bar{\eta}^a\bar{\eta}_i - \lambda^\alpha W_\alpha^{\beta i}\bar{\lambda}_\beta\bar{\eta}_i + \lambda^\alpha\eta_a M_\alpha^{aij}\bar{\eta}_i\bar{\eta}_j. \end{aligned} \quad (49)$$

This cocycle incorporates all the polyvector fields entering the right hand sides of the structure relations (48). For the mechanical systems without gauge symmetries and constraints these structure relations are absent and the corresponding Lagrange structure is determined by the first line in (49).

Due to the Lie algebra isomorphisms (46) and the identifications (47) the following conditions are pairwise equivalent:

$$\langle [\Omega_2], [\Omega_2] \rangle = 0 \quad \Leftrightarrow \quad \langle [L], [L] \rangle = 0 \quad \Leftrightarrow \quad \langle [P], [P] \rangle = 0. \quad (50)$$

In [15], a Lagrange structure was called *integrable* if all its Massey powers can be made zero. For mechanical systems this integrability condition boils down to vanishing of the Massey square of $[L]$. Relation (50) tells us that in the case of involutive systems of ODEs one can make one step further and reduce verification of the middle equality in (50) to verification of the rightmost one. This is an added reason for working with involutive normal forms, since the structure of a representative P is generally much simpler than that of L , as is seen from (49). In terms of representatives, the vanishing of the Massey square of $[P] = \nu^{-1}([L])$ amounts to the condition $[P, P] \in J$ or, explicitly,

$$[P, P] = U^\alpha \wedge R_\alpha - T_a S^a \quad (51)$$

for some vector fields U^α and bivector fields S^a .

A bivector field $P \in \Lambda^2(\mathbb{R}^n)$ is said to define a *weak Poisson structure* on \mathbb{R}^n if it satisfies the first two relations in (48) together with (51). Another name for P is P_∞ -structure [25]. Relation (51) is called the *weak Jacobi identity*. Given a weak Poisson structure P , a vector field V is called *weakly Hamiltonian* if it obeys the third relation in (48). The set of four polyvector fields (V, R, T, P) is referred to as a *weak Hamiltonian structure* on \mathbb{R}^n . If the right hand sides of relations (48) and (51) are equal to zero, then the adjective “weak” can be omitted. In this case, P is just a Poisson bivector, V and R_α ’s are the corresponding Poisson vector fields, and T_a ’s are Casimir functions for P . This is always true for mechanical systems without gauge symmetries and constraints.

Remark. The fact that any integrable Lagrange structure for the ordinary differential equations (1) without gauge symmetries and constraints is given simply by a V -invariant Poisson bivector P was found and explained in [7], [17]. In the opposite case of driftless systems, equations (1) assume the form of free differential algebra, if one regards x ’s and λ ’s as 0- and 1-forms, respectively. Any Lagrange structure for such equations are known to be equivalent to one determined by a bivector field [18]. Thus, our present conclusions are in line with the pervious results.

As we have seen in Sec. 4.2, the commutative algebra of physical observables $H_0^0(s_0)$ with values in local functions is isomorphic to a certain subquotient \mathcal{F} of the algebra $C^\infty(\mathbb{R}^n)$. Namely, let I denote the ideal of $C^\infty(\mathbb{R}^n)$ generated by the functions T_a . In view of the involutivity conditions (2) the gauge distribution R preserves I in the sense that $[R, I] \subset I$, and hence its action descends to the quotient $C^\infty(\mathbb{R}^n)/I$. By definition, the algebra \mathcal{F} is constituted by the R -invariant elements of $C^\infty(\mathbb{R}^n)/I$, cf. (17). In other words, a function $O \in C^\infty(\mathbb{R}^n)$ represents an observable $[O] \in \mathcal{F} \subset C^\infty(\mathbb{R}^n)/I$ if $[R_\alpha, O] \in I$ and two such functions O and O' represent the same observable, $[O] = [O']$, if $O - O' \in I$.

The weak Poisson structure $[P] \in \Lambda_J^2(\mathbb{R}^n)$ makes the commutative algebra \mathcal{F} into a Poisson algebra. The corresponding Poisson bracket is defined as a derived bracket [25] on representatives:

$$\{[O_1], [O_2]\}_P = [[P, O_1], O_2] \quad \forall [O_1], [O_2] \in \mathcal{F}.$$

Using the property of the Schouten bracket, one can easily verify that this bracket operation is well-defined and enjoys all the properties of a Poisson bracket: bilinearity, skew-symmetry, and the Jacobi identity. Furthermore, the Poisson algebra \mathcal{F} comes equipped with a derivation naturally induced by the drift V . Equating this derivation to the time derivative, we get the differential equation

$$D[O] = [[V, O]]$$

governing the evolution of a physical observable $[O] \in \mathcal{F}$. The nontrivial integrals of motion of the system (1) correspond then to the V -invariant observables. They constitute a Poisson subalgebra in \mathcal{F} , which, as a linear space, is isomorphic to the space of conservation laws $\Lambda_J^0(\mathbb{R}^n) \subset \mathcal{F}$. The last fact follows immediately from the definition of the space $\Lambda_J^0(\mathbb{R}^n)$.

In the absence of quantum anomalies, the Poisson algebra $(\mathcal{F}, \{\cdot, \cdot\})$ was shown to admit a consistent deformation quantization by means of a superextension of Kontsevich's formality theorem [6], [8]. The result of the deformation quantization is an associative $*$ -product in the space of quantum observables $\mathcal{F} \otimes \mathbb{C}[[\hbar]]$ together with a $*$ -product derivation \widehat{V} generating a one-parameter family of automorphisms of the quantum algebra $(\mathcal{F} \otimes \mathbb{C}[[\hbar]], *)$.

7. SUPERFIELD FORMULATION OF THE TOTAL BRST CHARGE

The weak Hamiltonian structure discussed in the previous section admits a nice BRST description in terms of generating functions [9]. Let us briefly recall its main details. Given a weakly Hamiltonian system (V, R, T, P) , the phase space \mathbb{R}^n of coordinates x^i is extended by the odd variables η_a and c^α called the ghosts: one η for each constraint T and one c for each gauge symmetry generator R . Denoting all the variables by $\phi^A = (x^i, \eta_a, c^\alpha)$, one then redoubles their number by introducing the dual variables $\phi_B^* = \{x_i^*, \eta^a, c_\alpha^*\}$ with opposite Grassmann parities. These are also called ghosts. The superspace W coordinatized by ϕ^A and ϕ_A^* is endowed with the canonical antisymplectic structure defined by the following antibrackets (odd Poisson brackets):

$$(\phi^A, \phi^B) = 0, \quad (\phi_A^*, \phi^B) = \delta_A^B, \quad (\phi_A^*, \phi_B^*) = 0.$$

Besides the Grassmann parity, all the variables carry three additional \mathbb{Z} -gradings, which are called, respectively, the ghost number, resolution degree and momentum degree⁶:

$$\begin{aligned} \text{gh}(x^i) &= 0, & \text{gh}(\eta^a) &= -1, & \text{gh}(c^\alpha) &= 1, & \text{gh}(\phi_A^*) &= 1 - \text{gh}(\phi^A), \\ \text{deg}(x^i) &= \text{deg}(x_i^*) = \text{deg}(\eta^a) = \text{deg}(c^\alpha) &= 0, & \text{deg}(c_\alpha^*) &= \text{deg}(\eta_a) = 1, \\ \text{Deg}(\phi^A) &= 0, & \text{Deg}(\phi_B^*) &= 1. \end{aligned} \tag{52}$$

In the absence of fermionic degrees of freedom (all x 's are even) the Grassmann parity and the ghost number are compatible in the usual sense:

$$\epsilon(\phi^A) = \text{gh}(\phi^A), \quad \epsilon(\phi_B^*) = \text{gh}(\phi_B^*) \pmod{2}.$$

⁶In [6], the momentum degree was referred to as the polyvector degree.

Now all the structure relations associated to the weak Hamiltonian structure (V, R, T, P) are compactly encoded in the pair of master equations

$$(S, S) = 0, \quad (S, \Gamma) = 0, \quad (53)$$

where the generating functions S and Γ are subject to the following grading and boundary conditions:

$$\text{gh}(S) = 2, \quad \epsilon(S) = 0, \quad \text{Deg}(S) > 0,$$

$$\text{gh}(\Gamma) = 1, \quad \epsilon(\Gamma) = 1, \quad \text{Deg}(\Gamma) > 0,$$

$$S = \eta^* T_a(x) + \dot{x}_i R_\alpha^i(x) c^\alpha + \dot{x}_i \dot{x}_j P^{ij}(x) + \dots, \quad \Gamma = \dot{x}_i V^i(x) + \dots.$$

The dots in the last line refer to the terms of positive resolution degree. All these terms can be systematically found from the master equations (53) by means of homological perturbation theory with respect to the resolution degree [6]. As is seen, the bosonic function S incorporates all the ingredients of the weak Poisson structure: the phase-space constraints T , the gauge symmetry generators R , and the weak Poisson bivector P . The weakly Hamiltonian vector field V – the drift – enters the fermionic function Γ . Expanding the master equations (53) in powers of ghosts, one readily recovers the involutivity conditions (2), defining relations (48), (51) for a weak Hamiltonian structure, and the hierarchy of their differential consequences.

As with the total BRST charge Ω , the generating function S gives rise to an L_∞ -structure on the space A of functions of momentum degree zero. If $S = \sum_{m=1}^\infty S_m$ is the expansion of S with respect to the momentum degree, then the n -th structure map $L_n : A^{\otimes n} \rightarrow A$ is given by

$$L_n : a_1 \otimes a_2 \otimes \dots \otimes a_n \mapsto \{a_1, a_2, \dots, a_n\} = (\dots (S_n, a_1), a_2), \dots, a_n) \in A. \quad (54)$$

In particular, the second structure map defines the weak Poisson bracket

$$\{a, b\} = ((S_2, a), b) = -(-1)^{\epsilon(a)\epsilon(b)} \{b, a\} \quad (55)$$

satisfying the weak Jacobi identity

$$\begin{aligned} & (-1)^{\epsilon(a)\epsilon(c)} \{\{a, b\}, c\} + (-1)^{\epsilon(c)\epsilon(b)} \{\{c, a\}, b\} + (-1)^{\epsilon(b)\epsilon(a)} \{\{b, c\}, a\} \\ &= (-1)^{\epsilon(a)\epsilon(c)+1} (s_0(a, b, c) + (s_0 a, b, c) + (-1)^{\epsilon(a)} (a, s_0 b, c) + (-1)^{\epsilon(a)+\epsilon(b)} (a, b, s_0 c)) \end{aligned} \quad (56)$$

where $s_0 = (S_1, \cdot)$ is the classical BRST differential. The operator s_0 differentiates the weak Poisson bracket (55) by the graded Leibniz rule. If $S_n = 0$ for all $n > 2$, then the first master equation (53) implies that (55) is a usual Poisson bracket determined by the Poisson bivector P . For a nondegenerate P the classical BRST differential is given then (locally) by a Hamiltonian vector fields $s_0 = \{\Omega, \cdot\}$, with Ω being the usual BFV-BRST-charge [1].

Remark. Formulae (40)-(43) and (54)-(56) show a striking algebraic parallelism in the two BRST formalisms for non-variational systems. Notice, however, a difference in the symmetry properties of the multibrackets (40) and (54): the multibrackets associated to the total BRST charge Ω are graded symmetric, while those associated to the function S are graded skew-symmetric. Actually, there are two equivalent definitions of an L_∞ -algebra, in terms of symmetric and skew-symmetric multibrackets, and one may use either of them. Equivalence is established by the parity reversion functor, see [25, Remark 2.1] for details.

In the previous section, we have shown that any Lagrange structure compatible with an involutive system of ODEs defines and is defined by some weakly Hamiltonian structure. So, there is a perfect correspondence between both the pictures of one and the same dynamics. Our argumentation, however, was somewhat indirect and heavily relied on the structure of local BRST cohomology. Below, we are going to present a direct construction of the total BRST charge Ω by the generating functions S and Γ of a weakly Hamiltonian structure. To that end, we shall follow the elegant superfield approach proposed in quite a similar context by Damgaard and Grigoriev [5].

Consider the superspace $\mathbb{R}^{1|1}$ with one even coordinate t , identified with time, and one odd coordinate θ , the odd superpartner of t . The smooth maps from $\mathbb{R}^{1|1}$ to the antisymplectic space W are described by the superfields $\phi^A(t, \theta)$ and $\phi_A^*(t, \theta)$, which form an infinite dimensional superspace \mathcal{W} . The canonical antisymplectic structure on W induces then a canonical symplectic structure on \mathcal{W} . The latter is defined by the Poisson brackets

$$\{\phi^A(z), \phi^B(z')\} = 0, \quad \{\phi_A^*(z), \phi^B(z')\} = \delta_A^B \delta(z - z'), \quad \{\phi_A^*(z), \phi_B^*(z')\} = 0, \quad (57)$$

where $z = (t, \theta)$. Since $\theta^2 = 0$, each superfield contains a pair of component fields that are just functions of time:

$$\phi^A(t, \theta) = \phi_0^A(t) + \theta \phi_1^A(t), \quad \phi_A^*(t, \theta) = \phi_A^0(t) + \theta \phi_A^1(t).$$

If we set $\text{gh}(\theta) = 1$ and $\text{Deg}(\theta) = 0$, then the ghost number and momentum degree of the component fields are unambiguously determined by those of superfields (52). It is, however, imposable to prescribe a definite resolution degree to θ . The zero-components of superfields define a trajectory in the antisymplectic space W . Introducing the individual notation for their superpartners

$$\phi_1^A(t) = \{\eta^i(t), -\xi_a(t), -\lambda^\alpha(t)\}, \quad \phi_A^1(t) = \{\bar{x}_i(t), \bar{\eta}^a(t), \bar{c}_\alpha(t)\}$$

and making identifications

$$\phi_A^0(t) = \{x_i^*(t), \eta^a(t), c_\alpha^*(t)\} = \{\bar{\eta}_i(t), \bar{\xi}^a(t), \bar{\lambda}_\alpha(t)\},$$

we see that the set of component fields $\{\phi_0^A, \phi_1^A, \phi_B^0, \phi_B^1\}$ exactly coincides with the set of fields $\{\varphi^I, \bar{\varphi}_J\}$ from Sec. 2, including the distribution of the Grassmann parities, ghost numbers and momentum degrees. Furthermore, evaluating the Poisson brackets (57) for the component fields, one can find that they are identical to the Poisson brackets (6). This means that the infinite dimensional symplectic superspaces V and \mathcal{W} with the Poisson brackets (6) and (57) are actually isomorphic to each other.

Now we define an odd homomorphism h relating the antibracket on W with the Poisson bracket on \mathcal{W} . To any function $F(\phi, \phi^*)$ on W the homomorphism h assigns the local functional

$$h(F) = \int dt d\theta F(\phi(t, \theta), \phi^*(t, \theta)).$$

It is easy to check that h is indeed a homomorphism of Lie algebras, i.e.,

$$\{h(F), h(G)\} = h((F, G)).$$

The last property holds true even if one allows the functions F and G to depend on t and θ as parameters. We are going to apply this homomorphism to the function

$$Q(\phi, \phi^*, \theta) = S(\phi, \phi^*) + \theta \Gamma(\phi, \phi^*),$$

which is just a linear combination of the generating functions of weak Hamiltonian structure. It is clear that $\epsilon(Q) = 0$ and $\text{gh}(Q) = 1$. Regarding θ as an external odd parameter, one can see that the master equations (53) are equivalent to the single equation

$$(Q, Q) = 0. \quad (58)$$

Consider now the functional

$$\Omega = \int dt d\theta \left(\phi_A^* D \phi^A + Q(\phi(t, \theta), \phi^*(t, \theta), \theta) \right), \quad D \equiv -\theta \frac{\partial}{\partial t}. \quad (59)$$

It satisfies all the grading conditions for the total BRST charge and verification of the master equation yields

$$\{\Omega, \Omega\} = h((Q, Q)) + 2 \int dt d\theta D S = 2 \int dt \frac{d}{dt} S(\phi_0(t), \phi_0^*(t)) = 0.$$

Here we used the master equation (58), the obvious identity $D^2 = 0$, and the zero boundary conditions for the fields of positive momentum degree.

Integration by θ in (59) yields the total BRST charge as the functional of component fields:

$$\Omega = \int dt \left\{ (-1)^{\epsilon(\phi_0^A)+1} \phi_A^0 \dot{\phi}_0^A + \phi_1^A \frac{\partial S}{\partial \phi^A}(\phi_0, \phi^0) + \phi_A^1 \frac{\partial S}{\partial \phi_A^*}(\phi_0, \phi^0) + \Gamma(\phi_0, \phi^0) \right\}. \quad (60)$$

Expanding the last expression further in powers of ghosts, one can see that the functional Ω meets also the boundary condition for the total BRST charge associated with the involutive equations (1) and the compatible Lagrange structure (49). Thus, formula (59) establishes a desired correspondence between the generating functions of a weak Hamiltonian structure and the total BRST charge. Let us mention two special properties of the BRST charge (60). First, the functional (60) involves no more than the first derivatives of fields, and these derivatives enter the Ω in a very peculiar way. Second, the functional (60) is at most linear in ϕ_1^A and ϕ_A^1 . The existence of such a solution to the master $\{\Omega, \Omega\} = 0$ is not easily seen without resort to the superfield approach.

8. CONCLUSION

In this paper, we presented a detailed analysis of the local BRST cohomology for general mechanical systems brought to the involutive normal form. The term “general” means that (i) we do not restrict ourselves to Lagrangian or constrained Hamiltonian systems and (ii) any regular system of ODEs can be equivalently reformulated in the involutive form at the cost of introducing auxiliary variables. Starting from the involutive normal form, we describe all the relevant groups of local BRST cohomology listed at the end of Sec. 3. In particular, we have identified the groups $H^{(1)}(\delta|D)$ with certain subquotients (28) of the algebra of polyvector fields on the phase space of the system. Thus, an explicit evaluation of these groups for a given model reduces to the standard problem of differential geometry. The most notable homogeneous subgroups of $H^{(1)}(\delta|D)$ are those associated with the spaces of conservation laws, global symmetries and Lagrange structure. Using the results of Sec. 4, we establish a one-to-one correspondence between the spaces of integrable Lagrange structures and weakly Hamiltonian structures. Establishing of such a correspondence is a matter of principle; it is as fundamental for the general dynamics as the correspondence between the BV and BFV quantization methods in the particular case of variational systems. Although our consideration was restricted to the mechanical systems, we hope that the computational technique developed in this paper can also be used in field theory with a due account of space locality. Finally, we gave a direct superfield construction of the total BRST charge Ω by the generating functions of the weakly Hamiltonian structure, Eqs. (59), (60). This generalizes the construction of Ref. [5] for the BV master action in terms of the BRST charge and unitarizing Hamiltonian. In the view of the aforementioned correspondence between the Lagrange and weakly Hamiltonian structures,

it is natural to ask about the inverse construction of the generating functions S and Γ by the total BRST charge Ω . Such a construction exists indeed, and we are going to present it elsewhere.

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